

FK-DLR states of a quantum bose-gas

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Abstract

The paper focuses on infinite-volume bosonic states for a quantum particle system (a quantum gas) in \mathbb{R}^d . The kinetic energy part of the Hamiltonian is the standard Laplacian (with a Dirichlet's boundary condition at the border of a 'box'). The particles interact with each other through a two-body finite-range potential depending on the distance between them and featuring a hard core of diameter $\mathbf{r} > 0$. We introduce a class of so-called FK-DLR functionals containing all limiting Gibbs states of the system. In the next paper we will prove that any FK-DLR functional is shift-invariant, regardless of whether it is unique or not.

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1 Introduction. Limit-point Gibbs states and reduced density matrices

This paper and its sequel, [12], continues the works [5], [6] and [7]. We pursue two lines of study: (i) a working definition of an infinite-volume quantum Gibbs state for various types of quantum bosonic systems, and (ii) its justification, which we have chosen to be the shift-invariance property for two-dimensional Bose-gas; cf. [12].

1.1. The local Hamiltonian. The object of study in this paper is a quantum Bose-gas in a Euclidean space \mathbb{R}^d . The starting point of our analysis is a self-adjoint n -particle Hamiltonian, $H_{n,\Lambda}$, of the system in a finite ‘box’ Λ . Typically, Λ is represented by a cube $[-L, +L]^{\times d}$, of size $2L > 0$, centered at the origin and with edges parallel to the co-ordinate axes. (Other types of bounded domains in \mathbb{R}^d can also be incorporated.) The operator $H_{n,\Lambda}$ is given by

$$\begin{aligned} (H_{n,\Lambda}\phi_n)(\underline{x}_1^n) &= -\frac{1}{2} \sum_{1 \leq j \leq n} (\Delta_j \phi_n)(\underline{x}_1^n) \\ &+ \sum_{1 \leq j < j' \leq n} V(|x(j) - x(j')|) \phi_n(\underline{x}_1^n), \quad \underline{x}_1^n = \{x(1), \dots, x(n)\} \in (\mathbb{R}^d)^n \end{aligned} \quad (1.1.1)$$

and acts on functions $\phi_n \in L_2^{\text{sym},\mathbf{r}}(\Lambda^n)$. Here $L_2^{\text{sym},\mathbf{r}}(\Lambda^n)$ stands for the subspace in the Hilbert space $L_2(\Lambda^n) = L_2(\Lambda)^{\otimes n}$ formed by symmetric functions of variables $x(j)$, $1 \leq j \leq n$, constituting the argument \underline{x}_1^n , which vanish whenever

$$\min [|x(j) - x(j')|_{\text{Eu}} : 1 \leq j < j' \leq n] < \mathbf{r}.$$

(Here and below, $|x|_{\text{Eu}}$, or briefly $|x|$, stands for the Euclidean norm of $x \in \mathbb{R}^d$ whereas $|x|_{\text{m}}$ denotes the max-norm.) Parameter $\mathbf{r} > 0$ is fixed and represents the diameter of the hard core (see below). It is convenient to denote

$$\Lambda_{\mathbf{r}}^n = \left\{ \underline{x}_1^n = (x(1), \dots, x(n)) \in \Lambda^n : \min [|x(j) - x(j')| : 1 \leq j < j' \leq n] \geq \mathbf{r} \right\} \quad (1.1.2)$$

and identify $L_2^{\text{sym},\mathbf{r}}(\Lambda^n)$ with $L_2^{\text{sym}}(\Lambda_{\mathbf{r}}^n)$, the Hilbert space of square-integrable symmetric functions $\phi_n(\underline{x}_1^n)$ with support in $\Lambda_{\mathbf{r}}^n$.

Operator Δ_j in (1.1.1) acts as a Laplacian in the variable $x(j)$. Further, $V : r \in [\mathbf{r}, +\infty) \mapsto V(r) \in \mathbb{R}$ is a C^2 -function with a compact support, describing a two-body interaction potential depending upon the distance between particles. The value

$$\mathbf{R} = \inf [r > 0 : V(\tilde{r}) \equiv 0 \text{ for } \tilde{r} \geq r] \quad (1.1.3)$$

is called the interaction radius (or the interaction range). Formally speaking, we set: $V(r) = +\infty$ for $0 \leq r < \mathbf{r}$, conforming with the hard-core assumption. We also set:

$$-\overline{V}^- = \min [V(r) : \mathbf{r} \leq r \leq \mathbf{R}], \quad (1.1.4)$$

with $\overline{V}^- = 0$ for $V \geq 0$, and

$$\overline{V}^{(1)} = \max [|V'(r)| : \mathbf{r} \leq r \leq \mathbf{R}]. \quad (1.1.5)$$

The above assumptions upon the two-body potential V are valid throughout the whole paper.

For $n = 1$, the sum $\sum_{1 \leq j < j' \leq n}$ in Eqn (1.1.1) is suppressed, and $H_{n,\Lambda}$ is reduced to $-\frac{1}{2}\Delta$ in Λ . For $n = 0$, we formally define $H_{0,\Lambda} = 0$. In general, the term $-\frac{1}{2} \sum_{1 \leq j \leq n} (\Delta_j \phi_n)(\underline{x}_1^n)$ represents the kinetic energy part in the Hamiltonian, and the term $\sum_{1 \leq j < j' \leq n} V(|x(j) - x(j')|)$ the potential energy (as an operator, it is given as multiplication by this function). Note that if n is large enough (viz., $n\pi\Gamma((d+1)/2)\mathbf{r}^d \geq (2L)^d = \text{volume of } \Lambda$) then the expression for $H_{n,\Lambda}$ formally becomes infinite; consequently, we only care about the values of n such that the set $\Lambda_{\mathbf{r}}^n \neq \emptyset$.

Operator $H_{n,\Lambda}$ is determined by a boundary condition. More precisely, it is initially defined by the RHS of Eqn (1.1.1) as a symmetric operator on the set of C^2 -functions $\phi = \phi_n$ with the support in the interior of $\Lambda_{\mathbf{r}}^n$. A self-adjoint extension of this symmetric operator is determined by boundary conditions. On the set $\partial^{\text{hc}}\Lambda_{\mathbf{r}}^n$:

$$\begin{aligned} \partial^{\text{hc}}\Lambda_{\mathbf{r}}^n = \Big\{ \underline{x}_1^n = (x(1), \dots, x(n)) \in \Lambda_{\mathbf{r}}^n : \\ \min [|x(j) - x(j')| : 1 \leq j < j' \leq n] = \mathbf{r} \Big\} \end{aligned} \quad (1.1.6)$$

we take the Dirichlet boundary condition:

$$\phi(\underline{x}_1^n) = 0 \text{ for } \underline{x}_1^n \in \partial^{\text{hc}} \Lambda_{\mathbf{r}}^n. \quad (1.1.7)$$

On the other hand, the cube $\Lambda = [-L, L]^{\times d}$ also has an ‘outer’ border $\partial\Lambda = \{y \in \mathbb{R}^d : |y|_{\text{m}} = L\}$. Next, given $n \geq 2$, we define:

$$\partial^{\text{out}} \Lambda_{\mathbf{r}}^n = \left\{ \underline{x}_1^n = (x(1), \dots, x(n)) \in \Lambda_{\mathbf{r}}^n : \max [|x(j)|_{\text{m}} : 1 \leq j \leq n] = L \right\}. \quad (1.1.8)$$

Throughout the paper we consider Dirichlet’s boundary condition on $\partial^{\text{out}} \Lambda_{\mathbf{r}}^n$:

$$\phi_n(\underline{x}_1^n) = 0, \quad \underline{x}_1^n \in \partial^{\text{out}} \Lambda_{\mathbf{r}}^n. \quad (1.1.9)$$

Nevertheless, the methods of this work allow us to consider a broad class of boundary conditions, including Neumann’s (zero of the normal derivative) and elastic (where a linear combination of the values of the function and its normal derivative vanishes); periodic boundary conditions can also included. Considering various boundary conditions endeavors towards enhancing possible phase transitions; we intend to return to this question in a forthcoming work.

Under the above assumptions, operator $H_{n,\Lambda}$ is self-adjoint, bounded from below and has a pure point spectrum. Moreover, $\forall \beta \in (0, +\infty)$, the Gibbs operator $G_{\beta,n,\Lambda} = \exp [-\beta H_{n,\Lambda}]$ is a positive-definite trace-class operator in $L_2^{\text{sym}}(\Lambda_{\mathbf{r}}^n)$. The trace

$$\Xi_{\beta,n}(\Lambda) := \text{tr}_{L_2^{\text{sym}}(\Lambda_{\mathbf{r}}^n)} G_{\beta,n,\Lambda} \in (0, +\infty) \quad (1.1.10)$$

is called the n -particle partition function in Λ at the inverse temperature β . When n is large and $\Lambda_{\mathbf{r}}^n$ becomes empty, we set $G_{\beta,n,\Lambda}$ to be a zero operator with $\Xi(\beta, n, \Lambda) = 0$. This allows us to work with the grand canonical Gibbs ensemble. Namely, $\forall z \in (0, +\infty)$, the direct sum

$$G_{\beta,\Lambda} = \bigoplus_{n \geq 0} z^n G_{\beta,n,\Lambda} \quad (1.1.11)$$

determines a positive-definite trace-class operator in the bosonic Fock space

$$\mathcal{H}(\Lambda) = \bigoplus_{n \geq 0} L_2^{\text{sym}}(\Lambda_{\mathbf{r}}^n). \quad (1.1.12)$$

The quantity

$$\Xi_{z,\beta}(\Lambda) := \sum_{n \geq 0} z^n \Xi(\beta, n, \Lambda) = \text{tr}_{\mathcal{H}(\Lambda)} G_{\beta,\Lambda} \in (0, +\infty) \quad (1.1.13)$$

is called the grand canonical partition function in Λ at fugacity z and the inverse temperature β . Further, the operator

$$R_{\beta,\Lambda} = \frac{1}{\Xi_{z,\beta}(\Lambda)} G_{z,\beta,\Lambda} \quad (1.1.14)$$

is called the (grand-canonical) density matrix (DM) in Λ ; this is a positive-definite operator in $\mathcal{H}(\Lambda)$ of trace 1. Operator $R_{z,\beta,\Lambda}$ determines the Gibbs state (GS), i.e., a linear positive normalized functional $\varphi_{z,\beta,\Lambda}$ on the C^* -algebra $\mathfrak{B}(\Lambda)$ of bounded operators in $\mathcal{H}(\Lambda)$:

$$\varphi_{z,\beta,\Lambda}(A) = \text{tr}_{\mathcal{H}(\Lambda)}(AR_{z,\beta,\Lambda}), \quad A \in \mathfrak{B}(\Lambda). \quad (1.1.15)$$

The next object of interest is the reduced DM (in short, the RDM), in cube $\Lambda_0 \subset \Lambda$ centered at a point $c_0 = (c_0^1, \dots, c_0^d)$:

$$\Lambda_0 = [-L_0 + c_0^1, c_0^1 + L_0] \times \dots \times [-L_0 + c_0^d, c_0^d + L_0]. \quad (1.1.16)$$

We use the term RDM for the partial trace

$$R_{z,\beta,\Lambda}^{\Lambda_0} = \text{tr}_{\mathcal{H}(\Lambda \setminus \Lambda_0)} R_{z,\beta,\Lambda}; \quad (1.1.17)$$

it is based on the tensor-product representation $\mathcal{H}(\Lambda) = \mathcal{H}(\Lambda_0) \otimes \mathcal{H}(\Lambda \setminus \Lambda_0)$. Operator $R_{z,\beta,\Lambda}^{\Lambda_0}$ acts in $\mathcal{H}(\Lambda_0)$, is positive-definite and has trace 1. Moreover, the partial trace operation leads to an important compatibility property for RDMs: if cubes $\Lambda_1 \subset \Lambda_0 \subset \Lambda$ then

$$R_{z,\beta,\Lambda}^{\Lambda_1} = \text{tr}_{\mathcal{H}(\Lambda_0 \setminus \Lambda_1)} R_{z,\beta,\Lambda}^{\Lambda_0}. \quad (1.1.18)$$

The mnemonic here is that the upper indices Λ_0 and Λ_1 indicate ‘volumes’ that have been kept ‘free’ of the partial trace.

The main results of the present paper are valid $\forall z, \beta \in (0, \infty)$ under the condition

$$\bar{\rho} := z \exp(4\beta \bar{V} R^d / r^d) < 1, \quad (1.1.19)$$

becoming $z \in (0, 1)$ if the two-body potential $V \geq 0$.¹ Recall, $\mathbf{r} \in (0, \infty)$ is the diameter of the hard core and $\mathbf{R} \in (\mathbf{r}, \infty)$ the radius of interaction; cf. (1.1.2) and (1.1.4). To simplify the notation, we will omit the indices/arguments z and β whenever it does not lead to a confusion. A straightforward generalization of the above concepts can be done by including an external potential field induced by an external classical configuration (CC) $\mathbf{x}(\Lambda^c)$ represented by a finite or countable subset in the complement Λ^c such that

$$|\bar{x} - \bar{x}'| \geq \mathbf{r} \quad \forall \quad \bar{x}, \bar{x}' \in \mathbf{x}(\Lambda^c) \quad \text{with} \quad \bar{x} \neq \bar{x}'. \quad (1.1.20)$$

(In fact, it is the intersection $\Lambda^{(\mathbf{R})} \cap \mathbf{x}(\Lambda^c)$ that will matter, where $\Lambda^{(\mathbf{R})} = \{x \in \mathbb{R}^d : \text{dist}(x, \Lambda) \leq \mathbf{R}\}$. Here and below, dist stands for the Euclidean distance.) Viz., the Hamiltonian $H_{n, \Lambda | \mathbf{x}(\Lambda^c)}$ is given by

$$(H_{n, \Lambda | \mathbf{x}(\Lambda^c)} \phi_n)(\underline{x}_1^n) = (H_{n, \Lambda} \phi_n)(\underline{x}_1^n) + \sum_{1 \leq j \leq n} \sum_{\bar{x} \in \mathbf{x}(\Lambda^c)} V(|x(j) - \bar{x}|) \phi_n(\underline{x}_1^n) \quad (1.1.21)$$

and has all properties that have been listed above for $H_{n, \Lambda}$. This enables us to introduce the Gibbs operators $G_{n, \Lambda | \mathbf{x}(\Lambda^c)}$ and $G_{\Lambda | \mathbf{x}(\Lambda^c)}$, the partition functions $\Xi_n(\Lambda | \mathbf{x}(\Lambda^c))$ and $\Xi(\Lambda | \mathbf{x}(\Lambda^c))$, the DM $R_{\Lambda | \mathbf{x}(\Lambda^c)}$, the GS $\varphi_{\Lambda | \mathbf{x}(\Lambda^c)}$ and the RDMS $R_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}$ where $\Lambda_0 \subset \Lambda$.² Viz.,

$$G_{n, \Lambda | \mathbf{x}(\Lambda^c)} = \exp \left[-\beta H_{n, \Lambda | \mathbf{x}(\Lambda^c)} \right], \quad (1.1.22)$$

$$\Xi_n(\Lambda | \mathbf{x}(\Lambda^c)) := \text{tr}_{L_2^{\text{sym}}(\Lambda^n)} G_{n, \Lambda | \mathbf{x}(\Lambda^c)} \in (0, +\infty),$$

and

$$G_{\Lambda | \mathbf{x}(\Lambda^c)} = \bigoplus_{n \geq 0} z^n G_{n, \Lambda | \mathbf{x}(\Lambda^c)}, \quad \Xi(\Lambda | \mathbf{x}(\Lambda^c)) := \sum_{n \geq 0} z^n \Xi_n(\Lambda | \mathbf{x}(\Lambda^c)) = \text{tr}_{\mathcal{H}(\Lambda)} G_{\Lambda | \mathbf{x}(\Lambda^c)} \in (0, +\infty). \quad (1.1.23)$$

In the case of an empty exterior $\mathbf{x}(\Lambda^c) = \emptyset$, the argument $\mathbf{x}(\Lambda^c)$ is omitted.

We conclude Section 1.1 with the following remark. The Fock spaces $\mathcal{H}(\Lambda)$ and $\mathcal{H}(\Lambda_0)$ (see (1.1.12)) can be conveniently represented as $L_2(\mathcal{C}_{\mathbf{r}}(\Lambda))$ and

¹Condition (1.1.18) is needed to guarantee a ‘thermodynamic stability’ of the system; apparently, it should not prevent phase transitions when the dimension $d \geq 2$.

²Although the Hamiltonian $H_{n, \Lambda}$ and its derivatives $G_{n, \Lambda}$, G_{Λ} and so on, are particular examples of $H_{n, \Lambda | \mathbf{x}(\Lambda^c)}$, etc. (with $\mathbf{x}(\Lambda^c)$ being an empty configuration), we will now and again address this specific example individually, for its methodological significance.

$L_2(\mathcal{C}_r(\Lambda^0))$, respectively. Here and below, $\mathcal{C}(\Lambda)$ denotes the collection of finite (unordered) subsets $\mathbf{x} \subset \Lambda$ (including the empty set) with the Lebesgue–Poisson measure

$$d\mathbf{x} = \frac{1}{(\sharp \mathbf{x})!} \prod_{x \in \mathbf{x}} dx, \quad \sharp \mathbf{x} < \infty \quad \left(\text{with } \int_{\mathcal{C}(\Lambda)} d\mathbf{x} = \exp[\ell(\Lambda)] \right) \quad (1.1.24)$$

where ℓ is the Lebesgue measure on \mathbb{R}^d ,

and $\mathcal{C}_r(\Lambda)$ stands for the subset of $\mathcal{C}(\Lambda)$ formed by $\mathbf{x} \subset \Lambda$ with

$$\min \left[|x - x'| : x, x' \in \mathbf{x}, x \neq x' \right] \geq r; \quad (1.1.25)$$

cf. (1.1.20). Here and later on, the symbol \sharp is used for the cardinality of a given set. The same meaning is attributed to the notation $\mathcal{C}_r(\mathbb{R}^d)$ and $\mathcal{C}_r(\Lambda^c)$ (here we bear in mind finite or countable sets $\mathbf{x} \subset \mathbb{R}^d$ and $\mathbf{x}' \subset \Lambda^c$, respectively, obeying (1.1.25)). Therefore, condition (1.1.20) is equivalent to writing $\mathbf{x}(\Lambda^c) \in \mathcal{C}_r(\Lambda^c)$. Points \mathbf{x}, \mathbf{x}' are called, as before, classical configurations (CCs).

1.2. The thermodynamic limit. The shift-invariance property in two dimensions. The key concept of Statistical Mechanics is the thermodynamic limit; in the context of this work it is $\lim_{\Lambda \nearrow \mathbb{R}^d}$. The quantities and objects established as limiting points in the course of this limit are often referred to as infinite-volume ones (e.g., an infinite-volume RDM or GS). The existence and uniqueness of a limiting object is often interpreted as absence of a phase transition. On the other hand, a multitude of such objects (viz., depending on the boundary conditions for the Hamiltonian or the choice of external CCs) is treated as a sign of a phase transition.

However, there exists an elegant alternative where infinite-volume objects are identified in terms which, at least formally, do not invoke the thermodynamic limit. For classical systems, this is the DLR equations and for so-called quantum spin systems – the KMS construction. (The latter involves an infinite-volume dynamics, a concept that is not affected by absence or presence of phase transitions.) Unfortunately, the KMS construction is not directly available for the class of quantum systems under consideration, since the Hamiltonians $H_{n,\Lambda}$ and $H_{n,\Lambda|\mathbf{x}(\Lambda^c)}$ are not bounded.

In this paper we propose a construction generalising the classical DLR equation (see Section 2.3). A justification of this construction will be given

in Ref. [12] where we establish the shift-invariance property for the emerging objects (the RDMs and GSs) in dimension two (i.e., for $d = 2$).³ For reader's convenience, we state the results from [12] below, after we give the assertions of the present paper.

The symbol \square marks the end of an assertion. The first result claimed in this work is

Theorem 1.1. *Assume inequality (1.1.19). \forall cube Λ_0 (see Eqn (1.1.16)), the family of RDMs $\{R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}, \Lambda \nearrow \mathbb{R}^d\}$ is compact in the trace-norm operator topology in $\mathcal{H}(\Lambda_0)$, for any choices of CCs $\mathbf{x}(\Lambda^c) \in \mathcal{C}_r(\Lambda^c)$ (i.e., satisfying (1.1.20)). Any limit-point operator R^{Λ_0} for $\{R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}\}$ is a positive-definite operator in $\mathcal{H}(\Lambda_0)$ of trace 1. Furthermore, let $\Lambda_1 \subset \Lambda_0$ be a pair of cubes and $R^{\Lambda_1}, R^{\Lambda_0}$ be a pair of limit-point RDMs such that*

$$R^{\Lambda_1} = \lim_{k \rightarrow +\infty} R_{\Lambda(k)|\mathbf{x}(\Lambda(k)^c)}^{\Lambda_1} \text{ and } R^{\Lambda_0} = \lim_{k \rightarrow +\infty} R_{\Lambda(k)|\mathbf{x}(\Lambda(k)^c)}^{\Lambda_0} \quad (1.2.1)$$

for a sequence of cubes $\Lambda(l) = [-L(l), L(l)]^{\times d}$ where $l = 1, 2, \dots, L(l) \nearrow \infty$ and external CCs $\mathbf{x}(\Lambda(l)^c) \in \mathcal{C}_r(\Lambda(l)^c)$. Then R^{Λ_1} and R^{Λ_0} satisfy the compatibility property

$$R^{\Lambda_1} = \text{tr}_{\mathcal{H}(\Lambda_0 \setminus \Lambda_1)} R^{\Lambda_0}. \quad (1.2.2)$$

\square

In future, referring to external CCs $\mathbf{x}(\Lambda^c)$ and $\mathbf{x}(\Lambda(l)^c)$, we always assume that $\mathbf{x}(\Lambda^c) \in \mathcal{C}_r(\Lambda^c)$ and $\mathbf{x}(\Lambda(l)^c) \in \mathcal{C}_r(\Lambda(l)^c)$, that is, the condition (1.1.20) is satisfied.

The subsequent sections of the paper carry the proof of Theorem 1.1.

A direct corollary of Theorem 1.1 is the construction of a limit-point Gibbs state φ . To this end, it suffices to consider a countable family of cubes $\Lambda_0(l_0) = [-L_0 l_0, L_0 l_0]^{\times d}$ of side-length $2L_0 l_0$, where $L_0 \in (0, \infty)$ is fixed and $l_0 = 1, 2, \dots$, centered at the origin. By invoking a diagonal process, we can guarantee that, given a family of external CCs $\mathbf{x}(\Lambda^c)$, one can extract a sequence $\Lambda(l) \nearrow \mathbb{R}^d$ such that (i) \forall positive integer $l_0 \exists$ the trace-norm limit

$$R^{\Lambda_0(l_0)} = \lim_{l \rightarrow +\infty} R_{\Lambda(l)|\mathbf{x}(\Lambda(l)^c)}^{\Lambda_0(l_0)}. \quad (1.2.3)$$

³In the one-dimensional case ($d = 1$) our construction yields uniqueness of an infinite-volume RDM and GS. This follows from earlier results; cf. [10] and [11].

and (ii) for the limiting operators relation (1.2.1) holds true with $\Lambda_1 = \Lambda_0(l_1)$ and $\Lambda_0 = \Lambda_0(l_0)$ whenever $l_1 < l_0$. This enables us to define an infinite-volume Gibbs state φ by setting

$$\varphi(A) = \lim_{l \rightarrow \infty} \varphi_{\Lambda(l)}(A) = \text{tr}_{\mathcal{H}(\Lambda_0(l_0))}(AR^{\Lambda_0(l_0)}), \quad A \in \mathfrak{B}(\Lambda_0(l_0)). \quad (1.2.4)$$

More precisely, φ is a state of the quasilocal C^* -algebra $\mathfrak{B}(\mathbb{R}^d)$ defined as the norm-closure of the inductive limit $\mathfrak{B}^0(\mathbb{R}^d)$:

$$\mathfrak{B} = (\mathfrak{B}^0(\mathbb{R}^d))^{-}, \quad \mathfrak{B}^0(\mathbb{R}^d) = \text{ind} \lim_{\Lambda \nearrow \mathbb{R}^d} \mathfrak{B}(\Lambda). \quad (1.2.5)$$

Moreover, ϕ is determined by a family of finite-volume RDMs R^{Λ_0} acting in $\mathcal{H}(\Lambda_0)$ where $\Lambda_0 \subset \mathbb{R}^d$ is an arbitrary cube of the form (1.1.16) and satisfying the compatibility property (1.2.2).

As was said, in paper [12] we establish the property of shift-invariance of the limit-point Gibbs states φ when the dimension $d = 2$. Observe that \forall cube Λ_0 as in (1.1.16) and vector $s = (s^1, \dots, s^d) \in \mathbb{R}^d$, the Fock spaces $\mathcal{H}(\Lambda_0)$ and $\mathcal{H}(\mathbf{S}(s)\Lambda_0)$ are related through a pair of mutually inverse shift isomorphisms of Fock spaces

$$\mathbf{U}^{\Lambda_0}(s) : \mathcal{H}(\Lambda_0) \rightarrow \mathcal{H}(\mathbf{S}(s)\Lambda_0) \text{ and } \mathbf{U}^{\mathbf{S}(s)\Lambda_0}(-s) : \mathcal{H}(\mathbf{S}(s)\Lambda_0) \rightarrow \mathcal{H}(\Lambda_0).$$

Here $\mathbf{S}(s)$ stands for the shift isometry $\mathbb{R}^d \rightarrow \mathbb{R}^d$:

$$\mathbf{S}(s) : y \mapsto y + s, \quad y \in \mathbb{R}^d, \quad (1.2.6)$$

and $\mathbf{S}(s)\Lambda_0$ for the image of Λ_0 :

$$\begin{aligned} \mathbf{S}(s)\Lambda_0 = & [-L_0 + c_0^1 + s^1, s^1 + c_0^1 + L^0] \\ & \times \dots \times [-L_0 + c_0^d + s^d, s^d + c_0^d + L^0]. \end{aligned} \quad (1.2.7)$$

The isomorphisms $\mathbf{U}^{\Lambda_0}(s)$ and $\mathbf{U}^{\mathbf{S}(s)\Lambda_0}(-s)$ are determined by

$$\begin{aligned} (\mathbf{U}^{\Lambda_0}(s)\phi_n)(\underline{x}_1^n) &= \phi_n(\mathbf{S}(-s)\underline{x}_1^n), \quad \underline{x}_1^n \in (\mathbf{S}(s)\Lambda_0)^n, \quad \phi_n \in L_2^{\text{sym}}((\Lambda_0)_{\mathbf{r}}^n), \\ (\mathbf{U}^{\mathbf{S}(s)\Lambda_0}(-s)\phi_n)(\underline{x}_1^n) &= \phi_n(\mathbf{S}(s)\underline{x}_1^n), \quad \underline{x}_1^n \in (\Lambda_0)^n, \quad \phi_n \in L_2^{\text{sym}}((\mathbf{S}(s)\Lambda_0)_{\mathbf{r}}^n), \end{aligned} \quad (1.2.8)$$

where $n = 0, 1, \dots$

In two dimensions, the main result of [12] is

Theorem 1.2. *Suppose that $d = 2$ and the condition (1.1.19) is fulfilled. Assume in addition that the potential V satisfies*

$$\overline{V}^{(2)} = \max [|V''(r)| : \mathbf{r} \leq r \leq \mathbf{R}]. \quad (1.2.9)$$

Then any limit-point Gibbs state φ is shift-invariant: $\forall s = (\mathbf{s}^1, \mathbf{s}^2) \in \mathbb{R}^2$

$$\varphi(A) = \varphi(\mathbf{S}(s)A), \quad A \in \mathfrak{B}(\mathbb{R}^2). \quad (1.2.10)$$

Here $\mathbf{S}(s)A$ stands for the shift of the argument A : if $A \in \mathfrak{B}(\Lambda_0)$ where Λ_0 is a square $[-L_0 + \mathbf{c}_1, \mathbf{c}_1 + L_0] \times [-L_0 + \mathbf{c}_2, \mathbf{c}_2 + L_0]$ then

$$\mathbf{S}(s)A = \mathbf{U}^{\mathbf{S}(s)\Lambda_0}(-s)A\mathbf{U}^{\Lambda_0}(s) \in \mathfrak{B}(\mathbf{S}(s)\Lambda_0).$$

In terms of the RDMs R^{Λ_0} :

$$R^{\mathbf{S}(s)\Lambda_0} = \mathbf{U}^{\Lambda_0}(s)R^{\Lambda_0}\mathbf{U}^{\mathbf{S}(s)\Lambda_0}(-s). \quad (1.2.11)$$

□

1.3. The integral kernels of Gibbs' operators and RDMs. Let us return to a general value of dimension d . We will assume condition (1.1.19) without stressing this every time again. According to the adopted realization of the Fock space $\mathcal{H}(\Lambda)$ as $L_2(\mathcal{C}_{\mathbf{r}}(\Lambda))$ (see (1.1.24)), its elements are represented by functions $\phi_{\Lambda} : \mathbf{x}(\Lambda) \in \mathcal{C}_{\mathbf{r}}(\Lambda) \mapsto \phi_{\Lambda}(\mathbf{x}(\Lambda)) \in \mathbb{C}$, with

$$\int_{\mathcal{C}(\Lambda)} |\phi_{\Lambda}(\mathbf{x}(\Lambda))|^2 d\mathbf{x}(\Lambda) < \infty. \quad (1.3.1)$$

The space $\mathcal{H}(\Lambda_0)$ is described in a similar manner: here we will use a shorthand notation \mathbf{x}_0 and \mathbf{y}_0 instead of $\mathbf{x}(\Lambda_0), \mathbf{y}(\Lambda_0) \in \mathcal{C}_{\mathbf{r}}(\Lambda_0)$.

The first step in the proof of Theorems 1.1 is to reduce its assertions to statements about the integral kernels $F_{\Lambda}^{\Lambda_0}$, $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and F^{Λ_0} which define the RDMs $R_{\Lambda}^{\Lambda_0}$, $R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and their infinite-volume counterpart R^{Λ_0} ; we call these kernels RDMKs for short. Indeed, $R_{\Lambda}^{\Lambda_0}$, $R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and R^{Λ_0} are integral operators:

$$(R_{\Lambda}^{\Lambda_0}\phi_{\Lambda})(\mathbf{x}_0) = \int_{\mathcal{C}_{\mathbf{r}}(\Lambda)} F_{\Lambda}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)\phi_{\Lambda}(\mathbf{y}_0)d\mathbf{y}_0, \quad (1.3.2)$$

$$\left(R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}\phi_\Lambda\right)(\mathbf{x}_0) = \int_{\mathcal{C}_\mathbf{r}(\Lambda)} F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \phi_\Lambda(\mathbf{y}_0) d\mathbf{y}_0 \quad (1.3.3)$$

and

$$(R^{\Lambda_0}\phi_\Lambda)(\mathbf{x}_0) = \int_{\mathcal{C}_\mathbf{r}(\Lambda)} F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \phi_\Lambda(\mathbf{y}_0) d\mathbf{y}_0. \quad (1.3.4)$$

The RDMKs $F_\Lambda^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ and $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ – and ultimately $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ – admit an FK representation providing a basis for the future analysis. Here we state properties of these kernels in Theorems 1.3 where we adopt a setting from Theorem 1.1:

Theorem 1.3. *\forall pair of cubes $\Lambda_0 \subset \Lambda$ and CCs $\mathbf{x}(\Lambda^c) \in \mathcal{C}_\mathbf{r}(\Lambda^c)$, the family of RDMKs $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ is compact in the space of continuous functions $C^0(\mathcal{C}_\mathbf{r}(\Lambda_0) \times \mathcal{C}_\mathbf{r}(\Lambda_0))$. Any limit-point function*

$$(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{C}_\mathbf{r}(\Lambda_0) \times \mathcal{C}_\mathbf{r}(\Lambda_0) \mapsto F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \quad (1.3.5)$$

determines a positive-definite operator R^{Λ_0} in $\mathcal{H}(\Lambda_0)$ of trace 1 (a limit-point RDM). Furthermore, let $\Lambda_1 \subset \Lambda_0$ be a pair of cubes and $F^{\Lambda_1}, F^{\Lambda_0}$ a pair of limit-point RDMKs such that

$$F^{\Lambda_1} = \lim_{k \rightarrow +\infty} F_{\Lambda(l)}^{\Lambda_1} \text{ and } F^{\Lambda_0} = \lim_{k \rightarrow +\infty} F_{\Lambda(l)|\mathbf{x}(\Lambda(l)^c)}^{\Lambda_0} \quad (1.3.6)$$

in $C^0(\mathcal{C}_\mathbf{r}(\Lambda_0) \times \mathcal{C}_\mathbf{r}(\Lambda_0))$ for a sequence of cubes $\Lambda(l) \nearrow \mathbb{R}^d$ and external CCs $\mathbf{x}(\Lambda(l)^c)$. Then the corresponding limit-point RDMs R^{Λ_1} and R^{Λ_0} obey (1.2.2). \square

Theorem 1.3 implies Theorem 1.1 with the help of Lemma 1.5 from [5] (going back to Lemma 1 in [10]). Therefore we focus on the proof of Theorems 1.3. In fact, we will establish the properties for more general objects – FK-DLR functionals. The rest of the paper is organized as follows. In Section 2.1 we introduce the FK-representation for the RDMKs $F_\Lambda^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ and $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ and in Section 2.2 the FK-representation for their infinite-volume counterparts $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$. On the basis of this representation we define the class of FK-DLR states (more generally, FK-DLR functionals) and state Theorems 2.1 and 2.2 extending the assertions of Theorems 1.4 and 1.5 to this class. Section 3 contains the proofs.

2 The FK representation and the FK-DLR equation

2.1. The background of the FK-representation. The symbol \triangle marks below the end of a definition. We begin with three definitions (see Definitions 2.1.1–2.1.3). In part, these definitions repeat pieces of papers [5], [6], [7].

Definition 2.1.1. (*Path spaces.*) As above, x, y stands for points in \mathbb{R}^d , $\underline{x} = \underline{x}_1^n = \{x(1), \dots, x(n)\}$ and $\underline{y} = \underline{y}_1^n = \{y(1), \dots, y(n)\}$ for points in Λ^n . Next, $\gamma = \gamma_n$ denotes a permutation of the n th order, $\gamma \underline{y} = \{y(\gamma(1)), \dots, y(\gamma(n))\}$ stands for the vector with permuted entries and $\mathbf{x}(\Lambda)$ for a point in $\mathcal{C}(\Lambda)$ (i.e., a finite subset of Λ). Furthermore, we will use the following system of notation:

(i) $\overline{\mathcal{W}}^{k\beta}(x, y)$ – the space of continuous paths $\overline{\omega} = \overline{\omega}_{x,y} : [0, k\beta] \rightarrow \mathbb{R}^d$ of time-length $k\beta$ (the parameter k is called the time-length multiplicity), with $\overline{\omega}^*(0) = x, \overline{\omega}^*(k\beta) = y$, where $k = 1, 2, \dots$;

(ii) $\overline{\mathcal{W}}^*(x, y) = \bigcup_{k \geq 1} \overline{\mathcal{W}}^{k\beta}(x, y)$ – the space of continuous paths $\overline{\omega}^* = \overline{\omega}_{x,y}^* : [0, \beta] \rightarrow \mathbb{R}^d$ of a variable time-length $k\beta$, with $\overline{\omega}^*(0) = x, \overline{\omega}^*(k\beta) = y$;

(iii) $\mathcal{W}^*(x) = \overline{\mathcal{W}}^*(x, x)$ – the space of loops (closed paths) $\omega^* = \omega_x^*$ with $\omega^*(0) = \omega^*(\beta) = x$;

(iv) $\overline{\mathcal{W}}^*(\underline{x}, \underline{y}) = \prod_{1 \leq j \leq n} \overline{\mathcal{W}}^*(x(j), y(j))$ – the space of (ordered) path collections $\overline{\Omega}^* = \{\overline{\omega}^*(1), \dots, \overline{\omega}^*(n)\}$ where $\overline{\omega}^*(j) \in \overline{\mathcal{W}}^*(x(j), y(j))$;

(v) $\underline{\mathcal{W}}^*(\underline{x}, \underline{y}) = \bigcup_{\gamma_n} \overline{\mathcal{W}}^*(\underline{x}, \gamma_n \underline{y})$ – the space of path collections $\overline{\Omega}^*$ with permuted endpoints;

(vi) $\mathcal{W}^*(\mathbf{x}) = \prod_{x \in \mathbf{x}} \mathcal{W}^*(x)$ – the space of loop collections $\Omega^*(\mathbf{x}) = \{\omega^*(x), x \in \mathbf{x}\}$ (Ω^* for short) with a given (finite) initial/end-point CC $\mathbf{x} \in \mathcal{C}_{\mathbf{r}}(\mathbb{R}^d)$, where $\omega(x) \in \mathcal{W}^*(x)$;

(vii) $\mathcal{W}^*(\Lambda) = \bigcup_{\mathbf{x} \in \mathcal{C}_{\mathbf{r}}(\Lambda)} \mathcal{W}^*(\mathbf{x})$ – the space of loop collections $\Omega^* = \Omega^*(\Lambda) = \{\omega^*(x), x \in \mathbf{x}\}$ with various initial/end-point CCs $\mathbf{x} = \mathbf{x}(\Lambda) \in \mathcal{C}_{\mathbf{r}}(\Lambda)$. Sometimes it will be helpful to stress that an element $\Omega^* \in \mathcal{W}^*(\Lambda)$ is a pair $[\mathbf{x}(\Lambda), \Omega^*(\mathbf{x}(\Lambda))]$ where $\Omega^*(\mathbf{x}(\Lambda)) \in \mathcal{W}^*(\mathbf{x}(\Lambda))$ and treat a loop $\omega^*(x) \in \mathcal{W}_x^*$ (or rather its shift $\mathbf{S}(-x)\omega^*(x) \in \mathcal{W}^*(0)$) as a ‘mark’ for point $x \in \mathbf{x}(\Lambda)$. (Here and below, the loop $\mathbf{S}(s)\omega^*$ is defined by $(\mathbf{S}(s)\omega^*)(\mathbf{t}) = \omega^*(\mathbf{t}) + s$, $s \in \mathbb{R}^d$, $\mathbf{t} \in [0, \beta k(\omega^*)]$.) Such a view is useful when we work with probability measures (PMs) on $\mathcal{W}^*(\Lambda)$: in the probabilistic terminology these

PMs represent two-dimensional random marked point processes (RMPPs) in Λ with marks from $\mathcal{W}^*(0)$, the space of loops starting and finishing at 0.

An element $\bar{\Omega}^*$ from $\bar{\mathcal{W}}^*(\underline{x}, \underline{y})$ is called a path configuration (PC), with the initial/terminal CCs $\underline{x}, \underline{y}$. Likewise, an element $\Omega^* \in \mathcal{W}^*(\Lambda)$ is called a loop configuration (LC) over Λ ; if $\Omega^* \in \mathcal{W}^*(\mathbf{x}(\Lambda))$, we say that $\mathbf{x}(\Lambda)$ is the initial CC for Ω^* . The time-length multiplicity of a path $\bar{\omega}^* \in \bar{\mathcal{W}}_{x,y}$ is denoted by $k(\bar{\omega}^*)$. The next series of definitions is introduced for a fixed $\mathbf{t} \in [0, \beta]$. Namely, given a path $\bar{\omega}^* \in \bar{\mathcal{W}}_{x,y}$, we call the set

$$\{\bar{\omega}^*(\bar{l}\beta + \mathbf{t}), \bar{l} = 0, \dots, k(\bar{\omega}^*) - 1\} \subset \mathbb{R}^d$$

the \mathbf{t} -section of $\bar{\omega}^*$ and denote it by $\{\bar{\omega}^*\}(\mathbf{t})$. Next, given a PC $\bar{\Omega}^* = \{\omega^*(1), \dots, \omega^*(n)\} \in \bar{\mathcal{W}}_{\underline{x}, \underline{y}}^*$, the \mathbf{t} -section for $\bar{\Omega}^*$ is defined as the union

$$\{\bar{\Omega}^*\}(\mathbf{t}) = \bigcup_{1 \leq j \leq n} \bar{\omega}^*(j, \mathbf{t})$$

where $\{\bar{\omega}^*(j)\}(\mathbf{t}) = \{\bar{\omega}^*(j, \bar{l}(j)\beta + \mathbf{t}), 0 \leq \bar{l}(j) < k(\bar{\omega}^*(j))\}$ is the \mathbf{t} -section for path $\bar{\omega}^*(j) \in \bar{\Omega}^*$ (thus, $\{\bar{\Omega}^*\}(\mathbf{t})$ again is a subset of \mathbb{R}^d). Likewise, given a LC $\Omega^* = \{\omega^*(x), x \in \mathbf{x}(\Lambda)\} \in \mathcal{W}^*(\mathbf{x}(\Lambda))$, the set

$$\{\Omega^*\}(\mathbf{t}) = \bigcup_{1 \leq j \leq n} \{\omega^*(x)\}(\mathbf{t}) \subset \mathbb{R}^d$$

is called the \mathbf{t} -section of Ω^* . Here $\{\omega^*(x)\}(\mathbf{t}) = \{\omega^*(x, l\beta + \mathbf{t}), 0 \leq l < k(\omega^*(x)), x \in \mathbf{x}(\Lambda)\} \subset \mathbb{R}^d$ is the \mathbf{t} -section of loop $\omega^*(x)$. The concept of a \mathbf{t} -section plays a subsidiary rôle in this paper but becomes essential in [12].

Similar definitions and terms will be used for a cube $\Lambda_0 \subset \Lambda$ or the set-theoretical difference $\Lambda \setminus \Lambda_0$.

All path/loop spaces $\bar{\mathcal{W}}^{k\beta}(x, y)$, $\bar{\mathcal{W}}^*(x, y)$, $\mathcal{W}^*(x)$, $\bar{\mathcal{W}}^*(\underline{x}, \underline{y})$, $\underline{\mathcal{W}}^*(\underline{x}, \underline{y})$, $\mathcal{W}^*(\mathbf{x}(\Lambda))$, $\mathcal{W}^*(\Lambda)$ from (i)–(vii) contain subsets $\bar{\mathcal{W}}_{\mathbf{r}}^{k\beta}(x, y)$, $\bar{\mathcal{W}}_{\mathbf{r}}^*(x, y)$, $\mathcal{W}_{\mathbf{r}}^*(x)$, $\bar{\mathcal{W}}_{\mathbf{r}}^*(\underline{x}, \underline{y})$, $\underline{\mathcal{W}}_{\mathbf{r}}^*(\underline{x}, \underline{y})$, $\mathcal{W}_{\mathbf{r}}^*(\mathbf{x}(\Lambda))$ and $\mathcal{W}_{\mathbf{r}}^*(\Lambda) = \bigcup_{\mathbf{x}(\Lambda) \in \mathcal{C}_{\mathbf{r}}(\Lambda)} \mathcal{W}_{\mathbf{r}}^*(\mathbf{x}(\Lambda))$ extracted by the condition that $\forall \mathbf{t} \in [0, \beta]$ no two distinct points in the \mathbf{t} -section lie at a Euclidean distance $\leq a$. In other words, all sections $\{\bar{\omega}^*\}(\mathbf{t})$, $\{\bar{\Omega}^*\}(\mathbf{t})$, $\{\Omega^*\}(\mathbf{t})$ are (finite) CCs lying in $\mathcal{C}_{\mathbf{r}}(\mathbb{R}^d)$. \triangle

Definition 2.1.2. (*Path measures.*) The spaces introduced in Definition 2.1.1 are equipped with standard sigma-algebras (generated by cylinder subsets and operations on them). We consider various measures on these sigma-algebras:

- (i) $\bar{\mathbb{P}}_{x,y}^{k\beta}$ – the (non-normalized) measure on $\bar{\mathcal{W}}_{x,y}^{k\beta}$ (the Wiener bridge of time-length $k\beta$), with $\bar{\mathbb{P}}_{x,y}^{k\beta}(\bar{\mathcal{W}}_{x,y}^{k\beta}) = (2\pi k\beta)^{-1} \exp \left[-|x-y|^2/(2k\beta) \right]$;
- (ii) $\bar{\mathbb{P}}_{x,y}^*$ – the sum-measure $\sum_{k \geq 1} \bar{\mathbb{P}}_{x,y}^{k\beta}$ on $\bar{\mathcal{W}}_{x,y}^*$;
- (iii) $\mathbb{P}_x^* = \bar{\mathbb{P}}_{x,x}^*$ – the sum-measure $\sum_{k \geq 1} \bar{\mathbb{P}}_x^{k\beta}$ on \mathcal{W}_x^* ;
- (iv) $\bar{\mathbb{P}}_{\underline{x},\underline{y}}^* = \prod_{1 \leq j \leq n} \bar{\mathbb{P}}_{x(j),y(j)}^*$ – the product-measure on $\bar{\mathcal{W}}_{\underline{x},\underline{y}}^*$ (a vector Wiener bridge) under which the components $\omega^*(j)$ are independent;
- (v) $\underline{\mathbb{P}}_{\underline{x},\underline{y}}^* = \sum_{\gamma_n} \bar{\mathbb{P}}_{\underline{x},\gamma_n \underline{y}}^*$ – the sum-measure on $\underline{\mathcal{W}}_{\underline{x},\underline{y}}^*$;
- (vi) $\mathbb{P}_{\mathbf{x}}^* = \prod_{x \in \mathbf{x}} \mathbb{P}_x^*$ – the product-measure on $\mathcal{W}^*(\mathbf{x})$;
- (vii) $d\Omega^*(\Lambda) = d\mathbf{x}(\Lambda) \times \mathbb{P}_{\mathbf{x}(\Lambda)}^*(d\Omega^*)$ – the measure on $\mathcal{W}^*(\Lambda)$ where $d\mathbf{x}(\Lambda)$ is the Lebesgue–Poisson measure on $\mathcal{C}(\Lambda)$ (cf. (1.3.2)). Sometimes we will write $d^\Lambda \mathbf{x}(\Lambda)$ and $d^\Lambda \Omega^*(\Lambda)$ in order to stress the dependence upon Λ (in particular, the notation $d^{\Lambda_0} \Omega_0^*$ will be used); other types of sets (the difference of two cubes) will also be employed. \triangle

Definition 2.1.3. (*Energy-related functionals.*) Given a path $\bar{\omega}^* \in \bar{\mathcal{W}}_{\mathbf{r}}^*(x, y)$, the functional $h(\bar{\omega}^*)$ is defined by

$$\begin{aligned} h(\bar{\omega}^*) &= \int_0^\beta dt \sum_{0 \leq l < l' < k(\bar{\omega}^*)} V(|\bar{\omega}^*(t + l\beta) - \bar{\omega}^*(t + l'\beta)|) \\ &= \int_0^\beta dt E(\{\bar{\omega}^*\}(t)). \end{aligned} \quad (2.1.1)$$

Here, for a given finite CC $\mathbf{z} \in \mathcal{C}_{\mathbf{r}}(\mathbb{R}^d)$, we set:

$$E(\mathbf{z}) = \frac{1}{2} \sum_{(z, z') \in \mathbf{z} \times \mathbf{z}} V(|z - z'|). \quad (2.1.2)$$

The quantity $h(\bar{\omega})$ can be interpreted as an energy of path $\bar{\omega}$.

The energy of interaction between two paths, $\bar{\omega}^* \in \bar{\mathcal{W}}_{\mathbf{r}}^*(x, y)$ and $\bar{\omega}'^* \in \bar{\mathcal{W}}_{\mathbf{r}}^*(x', y')$, is determined by

$$\begin{aligned} h(\bar{\omega}^*, \bar{\omega}'^*) &= \int_0^\beta dt \sum_{0 \leq l < k(\bar{\omega}^*)} \sum_{0 \leq l' < k(\bar{\omega}'^*)} V(|\bar{\omega}^*(t + l\beta) - \bar{\omega}'^*(t + l'\beta)|) \\ &= \int_0^\beta dt E(\{\bar{\omega}^*\}(t) \parallel \{\bar{\omega}'^*\}(t)). \end{aligned} \quad (2.1.3)$$

Here, for given pair of CCs $\mathbf{z}, \mathbf{z}' \in \mathcal{C}_{\mathbf{r}}(\mathbb{R}^d)$, such that $\mathbf{z} \cup \mathbf{z}' \in \mathcal{C}_{\mathbf{r}}(\mathbb{R}^d)$, $\mathbf{z} \cap \mathbf{z}' = \emptyset$ and at least one of them is finite, we set:

$$E(\mathbf{z}||\mathbf{z}') = \sum_{(z, z') \in \mathbf{z} \times \mathbf{z}'} V(|z - z'|). \quad (2.1.4)$$

The definitions (2.1.1) and (2.1.3) holds for loops as well, obviously.

Next, for a PC $\overline{\Omega}^* = \{\overline{\omega}^*(1), \dots, \overline{\omega}^*(n)\} \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\underline{x}, \underline{y})$ and a LC $\Omega^* = \{\omega^*(x)\} \in \mathcal{W}_{\mathbf{r}}^*(\mathbf{x}(\Lambda))$, the energy $h(\overline{\Omega}^*)$ of $\overline{\Omega}^*$ and the energy $h(\Omega^*)$ of Ω^* are defined as

$$h(\overline{\Omega}^*) = \sum_{1 \leq j \leq n} h(\overline{\omega}^*(j)) + \sum_{1 \leq j < j' \leq n} h(\overline{\omega}^*(j), \overline{\omega}^*(j')) \quad (2.1.5)$$

and

$$h(\Omega^*) = \sum_{x \in \mathbf{x}(\Lambda)} h(\omega^*(x)) + \frac{1}{2} \sum_{x, x' \in \mathbf{x}(\Lambda): x \neq x'} h(\overline{\omega}^*(x), \overline{\omega}^*(x')). \quad (2.1.6)$$

We will also need the energy for various combined collections of PCs, LCs and CCs. Viz., for $\overline{\Omega}^* \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\underline{x}, \underline{y})$ where $\underline{x}, \underline{y} \in \Lambda^n$ and $\Omega^* = \{\omega^*(x)\} \in \mathcal{W}_{\mathbf{r}}^*(\Lambda)$,

$$h(\overline{\Omega}^* \vee \Omega^*) = h(\overline{\Omega}^*) + h(\Omega^*) + h(\overline{\Omega}^* || \Omega^*) \quad (2.1.7)$$

where

$$h(\overline{\Omega}^* || \Omega^*) = \int_0^\beta d\mathbf{t} E(\overline{\Omega}^*(\mathbf{t}) || \Omega^*(\mathbf{t})). \quad (2.1.8)$$

Finally, for $\mathbf{x}(\Lambda^c) \in \mathcal{C}_{\mathbf{r}}(\Lambda^c)$,

$$h(\overline{\Omega}^* \vee \Omega^* | \mathbf{x}(\Lambda^c)) = h(\overline{\Omega}^* \vee \Omega^*) + h(\overline{\Omega}^* \vee \Omega^* || \mathbf{x}(\Lambda^c)) \quad (2.1.9)$$

where

$$h(\overline{\Omega}^* \vee \Omega^* || \mathbf{x}(\Lambda^c)) = \int_0^\beta d\mathbf{t} E(\{\overline{\Omega}^*\}(\mathbf{t}) \cup \{\Omega^*\}(\mathbf{t}) || \mathbf{x}(\Lambda^c)). \quad (2.1.10)$$

Finally, we introduce the functionals K , L and α_Λ , for path and LCs :

$$K(\overline{\Omega}^*) = \sum_{\overline{\omega}^* \in \overline{\Omega}^*} k(\overline{\omega}^*), \quad K(\Omega^*) = \sum_{\omega^* \in \Omega^*} k(\omega^*), \quad (2.1.11)$$

and

$$L(\Omega^*) = \prod_{\omega^* \in \Omega^*} k(\omega^*). \quad (2.1.12)$$

The presence of Dirichlet's boundary conditions is manifested in the indicators

$$\alpha_\Lambda(\bar{\Omega}^*) = \prod_{\bar{\omega}^* \in \bar{\Omega}^*} \alpha_\Lambda(\bar{\omega}^*), \quad \alpha_\Lambda(\Omega^*) = \prod_{\omega^* \in \Omega^*} \alpha_\Lambda(\omega^*), \quad (2.1.13)$$

where

$$\alpha_\Lambda(\bar{\omega}^*) = \mathbf{1}\left(\bar{\omega}^*(\mathbf{t}) \in \Lambda \quad \forall \mathbf{t} \in [0, k(\bar{\omega}^*)\beta]\right). \quad (2.1.14)$$

\triangle

2.2. The FK-representation in a cube. As follows from well-known results about the operator H_Λ (see, e.g., [4], [9]), we have the following properties listed in Lemmas 2.2.1 and 2.2.2.

Lemma 2.2.1. *For a chosen external CC $\mathbf{x}(\Lambda^c)$ defining the self-adjoint operators $H_{n,\Lambda|\mathbf{x}(\Lambda^c)}$, the partition function $\Xi[\Lambda|\mathbf{x}(\Lambda^c)]$ (see (1.1.23)) admits the following representation:*

$$\begin{aligned} \Xi[\Lambda|\mathbf{x}(\Lambda^c)] &= \int_{\mathcal{W}_r^*(\Lambda)} d\Omega_\Lambda^* \alpha_\Lambda(\Omega_\Lambda^*) \\ &\quad \times \frac{z^{K(\Omega_\Lambda^*)}}{L(\Omega_\Lambda^*)} \exp \left[-h\left(\Omega_\Lambda^*|\mathbf{x}(\Lambda^c)\right) \right]. \end{aligned} \quad (2.2.1)$$

See Definitions 2.1.1(iii),(vi),(vii) and 2.1.2(iii),(vi),(vii).

Moreover, for the corresponding RDMK $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ (see (1.3.3), (1.3.4)) we have that for $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}_r(\Lambda_0)$ with $\# \mathbf{x}_0 = \# \mathbf{y}_0$:

$$\begin{aligned} F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) &= \int_{\mathcal{W}_r^*(\mathbf{x}_0, \mathbf{y}_0)} \mathbb{P}_{\mathbf{x}_0, \mathbf{y}_0}^* (d\underline{\Omega}_0^*) \chi^{\Lambda_0}(\underline{\Omega}_0^*) \alpha_\Lambda(\underline{\Omega}_0^*) z^{K(\underline{\Omega}_0^*)} \hat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\underline{\Omega}_0^*). \end{aligned} \quad (2.2.2)$$

Cf. Definitions 2.1.1(v) and 2.1.2(v). Here

$$\hat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\underline{\Omega}_0^*) = \frac{\hat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}[\Lambda \setminus \Lambda_0|\mathbf{x}(\Lambda^c)]}{\Xi[\Lambda|\mathbf{x}(\Lambda^c)]}, \quad \underline{\Omega}_0^* \in \mathcal{W}_r^*(\mathbf{x}_0, \mathbf{y}_0), \quad (2.2.3)$$

$\Xi[\Lambda|\mathbf{x}(\Lambda^c)]$ is defined as in (2.2.1) and

$$\begin{aligned} & \widehat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}[\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^c)] \\ &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d\Omega_{\Lambda \setminus \Lambda_0}^* \mathbf{1}\left(\Omega_{\Lambda \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda_0)\right) \chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda_0}^*) \\ & \quad \times \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda_0}^*) \frac{z^{K(\Omega_{\Lambda \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda \setminus \Lambda_0}^*)} \exp \left[-h\left(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \mathbf{x}(\Lambda^c)\right) \right]. \end{aligned} \quad (2.2.4)$$

Functionals K and L are as in (2.1.11), (2.1.12). Next, $\chi^{\Lambda_0}(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^*)$ stands for the indicator requiring that no path $\bar{\omega}^*$ or loop ω^* from the whole collection enters the cube Λ_0 at ‘control’ time points $l\beta$ with $1 \leq l < k$, where k equals $k(\bar{\omega}^*)$ or $k(\omega^*)$.

Namely, for a PC $\underline{\Omega}_0^* = \{\bar{\omega}^*(1), \dots, \bar{\omega}^*(n)\} \in \underline{\mathcal{W}}^*(\mathbf{x}_0, \mathbf{y}_0)$ where $\underline{x} = \{x(1), \dots, x(n)\}$, $\underline{y} = \{y(1), \dots, y(n)\} \in \mathcal{C}(\Lambda_0)$ and a LC $\Omega_{\Lambda \setminus \Lambda_0}^* = \{\omega^*(x), x \in \mathbf{x}_{\Lambda \setminus \Lambda_0}\}$ where $\mathbf{x}_{\Lambda \setminus \Lambda_0} \in \mathcal{C}(\Lambda)$,

$$\begin{aligned} & \chi^{\Lambda_0}(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^*) \\ &= \mathbf{1}\left(\bar{\omega}^*(j, l\beta) \in \mathbb{R}^d \setminus \Lambda \forall l = 1, \dots, k(\bar{\omega}^*(j)) - 1, 1 \leq j \leq n\right) \\ & \quad \times \mathbf{1}\left(\omega^*(x, l\beta) \in \mathbb{R}^d \setminus \Lambda \forall l = 1, \dots, k(\omega^*(x)) - 1, x \in \mathbf{x}(\Lambda)\right). \end{aligned} \quad (2.2.5)$$

Note that when $k(\bar{\omega}^*(j)) = 1$ or $k(\omega^*(x)) = 1$, the above indicator yields no restriction. \square

Mnemonicly, the notation $\widehat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}$ means the application of an indicator function χ^{Λ_0} in the corresponding integral, together with presence of a specific PC $\underline{\Omega}_0^*$ in the energy functional $h\left(\underline{\Omega}_0^* \vee \Omega^* | \mathbf{x}(\Lambda^c)\right)$. We can say that the quantity $\widehat{\Xi}^{\Lambda_0, \bar{\Omega}_0^*}[\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^c)]$ in (2.2.3) represents a restricted partition function in $\Lambda \setminus \Lambda_0$ in presence of a PC $\bar{\Omega}_0^*$ and in the potential field generated by an external CC $\mathbf{x}(\Lambda^c)$, with the restriction dictated by χ^{Λ_0} . We would like to note that $\widehat{\Xi}^{\Lambda_0, \bar{\Omega}_0^*}[\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^c)]$ is only one out of several types of partition functions that we will have to deal with in our analysis.

The aftermath of Lemma 2.1 is the emergence of a probability measure (PM), $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$, on the LC space $\mathcal{W}_{\mathbf{r}}^*(\Lambda)$ (i.e., an RMPP in Λ with marks from the loop space $\mathcal{W}_{\mathbf{r}}^*(0)$). More precisely, $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ is a PM on the standard (Borel) sigma-algebra $\mathfrak{W}(\Lambda)$ of subsets of $\mathcal{W}^*(\Lambda)$ supported by $\mathcal{W}_{\mathbf{r}}^*(\Lambda)$.

Definition 2.4. The PM $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ is given by the probability density function (PDF) $f_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*)$, $\Omega^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda)$, where

$$\begin{aligned} f_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*) &:= \frac{\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(d\Omega^*)}{d\Lambda\Omega^*} \\ &= \alpha_{\Lambda}(\Omega^*) \frac{z^{K(\Omega_{\Lambda}^*)}}{L(\Omega_{\Lambda}^*)} \frac{\exp \left[-h\left(\Omega_{\Lambda}^*|\mathbf{x}(\Lambda^c)\right) \right]}{\Xi[\Lambda|\mathbf{x}(\Lambda^c)]}, \end{aligned} \quad (2.2.6)$$

with partition function $\Xi[\Lambda|\mathbf{x}(\Lambda^c)]$ as in (1.1.21), (2.2.1). Furthermore, consider the restriction $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}|_{\mathfrak{W}_{\mathbf{r}}^*(\Lambda_0)}$ of $\mu_{\Lambda,(\Lambda^c)}$ to the sigma-algebra $\mathfrak{W}^*(\Lambda_0)$ (more precisely, to $(\mathcal{W}_{\mathbf{r}}^*(\Lambda_0), \mathfrak{W}^*(\Lambda_0))$). Here $\mathfrak{W}^*(\Lambda_0)$ is treated as a sigma-subalgebra of $\mathfrak{W}^*(\Lambda)$, through the map $\mathcal{W}^*(\Lambda) \rightarrow \mathcal{W}^*(\Lambda_0)$:

$$\Omega^* = \{\omega_x^*, x \in \mathbf{x}(\Lambda) \subset \Lambda\} \mapsto \Omega_{\Lambda_0}^* = \{\omega_x^* : x \in \mathbf{x}(\Lambda) \cap \Lambda_0\}.$$

Then $\mu_{\Lambda,(\Lambda^c)}$ is determined by the PDF

$$\begin{aligned} f_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*) &:= \frac{\mu_{\Lambda|\mathbf{x}(\Lambda^c)}|_{\mathcal{W}^*(\Lambda_0)}(d\Omega_0^*)}{d^{\Lambda_0}\Omega_0^*} \\ &= \alpha_{\Lambda}(\Omega_0^*) \frac{z^{K(\Omega_0^*)}}{L(\Omega_0^*)} \frac{\Xi^{\Lambda_0, \Omega_0^*}[\Lambda \setminus \Lambda_0|\mathbf{x}(\Lambda^c)]}{\Xi[\Lambda|\mathbf{x}(\Lambda^c)]}, \quad \Omega_0^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda_0). \end{aligned} \quad (2.2.7)$$

Here the numerator $\Xi^{\Lambda_0, \Omega_0^*}[\Lambda \setminus \Lambda_0|\mathbf{x}(\Lambda^c)]$ is given by

$$\begin{aligned} &\Xi^{\Lambda_0, \Omega_0^*}[\Lambda \setminus \Lambda_0|\mathbf{x}(\Lambda^c)] \\ &= \int_{\mathcal{W}_{\mathbf{r}}(\Lambda)} d^{\Lambda}\Omega_{\Lambda \setminus \Lambda_0}^* \mathbf{1}\left(\Omega_{\Lambda \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}(\Lambda \setminus \Lambda_0)\right) \\ &\quad \times \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda_0}^*) \frac{z^{K(\Omega_{\Lambda \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda \setminus \Lambda_0}^*)} \exp \left[-h\left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^*|\mathbf{x}(\Lambda^c)\right) \right]. \end{aligned} \quad (2.2.8)$$

One can say that quantity $\Xi^{\Lambda_0, \Omega_0^*}[\Lambda \setminus \Lambda_0|\mathbf{x}(\Lambda^c)]$ in (2.2.8) represents a partition function in $\Lambda \setminus \Lambda_0$ in the external field generated by the CC $\mathbf{x}(\Lambda^c)$, in presence of a LC Ω_0^* over Λ_0 . \triangle

Lemma 2.2 *The PM $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ satisfies the following property: $\forall \Lambda_0 \subset \Lambda$ and $\Omega_0^* \in \mathcal{W}^*(\Lambda_0)$, the PDF $f_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*)$ introduced in (2.2.7) has the form*

$$f_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*) = \mathbf{1}\left(\Omega_0^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda_0)\right) \alpha_{\Lambda}(\Omega_0^*) \frac{z^{K(\Omega_0^*)}}{L(\Omega_0^*)} q_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*). \quad (2.2.9)$$

Here $\forall \Lambda_0 \subseteq \Lambda' \subseteq \Lambda$, the functional $q_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*)$ admits the representation

$$\begin{aligned} q_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_{\Lambda \setminus \Lambda'}^*) \mathbf{1}\left(\Omega_{\Lambda \setminus \Lambda'}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda')\right) \\ &\quad \times \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda'}^*) \Xi^{\Lambda_0, \Omega_0^*} \left[\Lambda' \setminus \Lambda_0, \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right]. \end{aligned} \quad (2.2.10)$$

Moreover, for a given $\Omega_{\Lambda \setminus \Lambda'}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda')$, the conditional partition function $\Xi^{\Lambda_0, \Omega_0^*} \left[\Lambda' \setminus \Lambda_0, \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right]$ is defined in a manner similar to quantity $\Xi^{\Lambda_0, \Omega_0^*} [\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^c)]$ in (2.2.8):

$$\begin{aligned} &\Xi^{\Lambda_0, \Omega_0^*} \left[\Lambda' \setminus \Lambda_0, \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right] \\ &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d^{\Lambda} \Omega_{\Lambda' \setminus \Lambda_0}^* \mathbf{1}\left(\Omega_{\Lambda' \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda' \setminus \Lambda_0)\right) \alpha_{\Lambda}(\Omega_{\Lambda' \setminus \Lambda_0}^*) \\ &\quad \times \frac{z^{K(\Omega_{\Lambda' \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda' \setminus \Lambda_0}^*)} \exp \left[-h\left(\Omega_0^* \vee \Omega_{\Lambda' \setminus \Lambda_0}^* | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c)\right) \right]. \end{aligned} \quad (2.2.11)$$

Furthermore, the RDMK $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ can be written as an integral:

$$\begin{aligned} F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) &= \int_{\underline{\mathcal{W}}_{\mathbf{r}}^*(\mathbf{x}_0, \mathbf{y}_0)} \mathbb{P}_{\mathbf{x}_0, \mathbf{y}_0}^* (d\overline{\Omega}_0^*) \chi^{\Lambda_0}(\overline{\Omega}_0^*) \\ &\quad \times \alpha_{\Lambda}(\overline{\Omega}_0^*) z^{K(\overline{\Omega}_0^*)} \widehat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\overline{\Omega}_0^*). \end{aligned} \quad (2.2.12)$$

Here the functional $\widehat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\overline{\Omega}_0^*)$ admits the following representation: $\forall \Lambda_0 \subseteq \Lambda' \subseteq \Lambda$,

$$\begin{aligned} \widehat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\overline{\Omega}_0^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_{\Lambda \setminus \Lambda'}^*) \mathbf{1}\left(\Omega_{\Lambda \setminus \Lambda'}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda')\right) \\ &\quad \times \chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda'}^*) \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda'}^*) \widehat{\Xi}^{\Lambda_0, \overline{\Omega}_0^*} \left[\Lambda' \setminus \Lambda_0 | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right]. \end{aligned} \quad (2.2.13)$$

Moreover, in analogy with (2.2.11), for a given $\Omega_{\Lambda \setminus \Lambda'}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda')$,

$$\begin{aligned} &\widehat{\Xi}^{\Lambda_0, \overline{\Omega}_0^*} \left[\Lambda' \setminus \Lambda_0 | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right] \\ &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d^{\Lambda} \Omega_{\Lambda' \setminus \Lambda_0}^* \mathbf{1}\left(\Omega_{\Lambda' \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda' \setminus \Lambda_0)\right) \chi^{\Lambda_0}(\Omega_{\Lambda' \setminus \Lambda_0}^*) \\ &\quad \times \alpha_{\Lambda}(\Omega_{\Lambda' \setminus \Lambda_0}^*) \frac{z^{K(\Omega_{\Lambda' \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda' \setminus \Lambda_0}^*)} \exp \left[-h\left(\overline{\Omega}_0^* \vee \Omega_{\Lambda' \setminus \Lambda_0}^* | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c)\right) \right]. \end{aligned} \quad (2.2.14)$$

□

Note that the presence of terms $\exp \left[-h \left(\Omega_0^* \vee \Omega_{\Lambda' \setminus \Lambda_0}^* | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right) \right]$ and $\exp \left[-h \left(\underline{\Omega}_0^* \vee \Omega_{\Lambda' \setminus \Lambda_0}^* | \Omega_{\Lambda \setminus \Lambda'}^* \vee \mathbf{x}(\Lambda^c) \right) \right]$ in (2.2.11) and (2.2.14) implies the presence of the indicators $\mathbf{1} \left(\Omega_0^* \vee \Omega_{\Lambda' \setminus \Lambda_0}^* \vee \Omega_{\Lambda \setminus \Lambda'}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda) \right)$ and $\mathbf{1} \left((\underline{\Omega}_0^*, \Omega_{\Lambda' \setminus \Lambda_0}^* \vee \Omega_{\Lambda \setminus \Lambda'}^*) \in \underline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \Lambda \setminus \Lambda_0) \right)$.

In particular, for $\Lambda_0 = \Lambda'$, Eqns (2.2.10) and (2.2.13) take the form:

$$\begin{aligned} q_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}(\Omega_0^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d\mu_{\Lambda | \mathbf{x}(\Lambda^c)}(\Omega_{\Lambda \setminus \Lambda_0}^*) \mathbf{1}(\Omega_{\Lambda \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda_0)) \\ &\quad \times \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda_0}^*) \exp \left[-h \left(\Omega_0^* | \Omega_{\Lambda \setminus \Lambda_0}^* \vee \mathbf{x}(\Lambda^c) \right) \right] \end{aligned} \quad (2.2.15)$$

and

$$\begin{aligned} \hat{q}_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}(\underline{\Omega}_0^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda)} d\mu_{\Lambda | \mathbf{x}(\Lambda^c)}(\Omega_{\Lambda \setminus \Lambda_0}^*) \mathbf{1}(\Omega_{\Lambda \setminus \Lambda_0}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda_0)) \\ &\quad \times \chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda_0}^*) \alpha_{\Lambda}(\Omega_{\Lambda \setminus \Lambda_0}^*) \exp \left[-h \left(\underline{\Omega}_0^* | \Omega_{\Lambda \setminus \Lambda_0}^* \vee \mathbf{x}(\Lambda^c) \right) \right]. \end{aligned} \quad (2.2.16)$$

On the other hand, when $\Lambda' = \Lambda$, Eqn (2.2.13) coincides with (2.2.4).

We would like to stress here that the integral $\int d\mu_{\Lambda | \mathbf{x}(\Lambda^c)}(\Omega_{\Lambda \setminus \Lambda'}^*)$ in (2.2.9), (2.2.13) is taken in the variable $\Omega_{\Lambda \setminus \Lambda'}^*$ considered as an element of space $\mathcal{W}_{\mathbf{r}}^*(\Lambda)$. Likewise, the integral $\int d\Lambda \Omega_{\Lambda' \setminus \Lambda_0}^*$ in (2.2.11), (2.2.14) is taken in the variable $\Omega_{\Lambda' \setminus \Lambda_0}^*$ considered as an element of space $\mathcal{W}_{\mathbf{r}}^*(\Lambda)$.

Eqns (2.2.9)–(2.2.14) are called the FK-DLR equations in volume Λ .

2.3. The infinite-volume FK-DLR equations and RDMKs.

Infinite-volume versions of the RDMK arise when we mimic properties listed in Lemmas 2.1 and 2.2 by getting rid of the reference to the enveloping cube Λ (this includes the external CC $\mathbf{x}(\Lambda^c)$ and the functional α_{Λ} indicating Dirichlet's boundary condition). The first place to do so is the PM $\mu_{\Lambda | \mathbf{x}(\Lambda^c)}$; to this end we need to consider its infinite-volume analog $\mu_{\mathbb{R}^d}$ representing

an RMPP in the whole plane \mathbb{R}^d . Formally, $\mu_{\mathbb{R}^d}$ yields a PM on the sigma-algebra $\mathfrak{W}(\mathbb{R}^d)$ of subsets in $\mathcal{W}^*(\mathbb{R}^d)$. The space $\mathcal{W}^*(\mathbb{R}^d)$ is formed by pairs $[\mathbf{x}(\mathbb{R}^d), \Omega^*(\mathbf{x}(\mathbb{R}^d))]$ where $\mathbf{x}(\mathbb{R}^d)$ is a locally finite set in the plane and $\Omega^*(\mathbf{x}(\mathbb{R}^d))$ (in brief, $\Omega_{\mathbb{R}^d}^*$ or simply Ω^*) is a collection $\{\omega^*(x), x \in \mathbf{x}(\mathbb{R}^d)\}$ of loops $\omega^*(x) \in \mathcal{W}_x^*$. Alternatively, $\Omega^*(\mathbf{x}(\mathbb{R}^d)) \in \prod_{x \in \mathbf{x}(\Lambda)} \mathcal{W}_x^*$. Next, $\mathfrak{W}(\mathbb{R}^d)$ is

the sigma-algebra of subsets in $\mathcal{W}^*(\mathbb{R}^d)$ generated by the cylinder events. It is convenient to refer to $\mathfrak{W}(\mathbb{R}^d)$ as the smallest sigma-algebra containing all sigma-algebras $\mathfrak{W}(\Lambda)$ where Λ is a cube and $\mathfrak{W}(\Lambda)$ is formed by the inverse images of sets $A \in \mathfrak{W}_\Lambda$ under the maps

$$\mathbf{x}(\mathbb{R}^d) \mapsto \mathbf{x}(\Lambda), \quad \Omega_{\mathbb{R}^d}^* \mapsto \Omega_\Lambda^*.$$

Here $\mathbf{x}(\Lambda) = \mathbf{x}(\mathbb{R}^d) \cap \Lambda$ and, for $\Omega_{\mathbb{R}^d}^* = \{\omega^*(x), x \in \mathbf{x}(\mathbb{R}^d)\}$, the symbol Ω_Λ^* stands for the sub-collection $\{\omega^*(x), x \in \mathbf{x}(\Lambda)\}$.

To simplify technical aspects of the presentation, we will omit the reference to the initial CC $\mathbf{x}(\mathbb{R}^d)$ and write $\Omega_{\mathbb{R}^d}^* \in \mathcal{W}^*(\mathbb{R}^d)$ or $\Omega^* \in \mathcal{W}^*(\mathbb{R}^d)$ (given a LC Ω^* , the initial CC is uniquely determined and can be denoted by $\mathbf{x}(\Omega^*)$).

Furthermore, we will use the notation $\mathcal{W}^*(\Lambda^c)$ for the subset in $\mathcal{W}^*(\mathbb{R}^d)$ formed by LCs $\Omega_{\Lambda^c}^*$ with $\mathbf{x}(\Omega_{\Lambda^c}^*) \in \mathcal{C}_r(\Lambda^c)$. (We call such $\Omega_{\Lambda^c}^*$ a LC over Λ^c .)

Definition 2.5. We say that a PM $\mu = \mu_{\mathbb{R}^d}$ on $(\mathcal{W}^*(\mathbb{R}^d), \mathfrak{W}(\mathbb{R}^d))$ satisfies the (infinite-volume) FK-DLR equations if the restriction $\mu|_{\mathfrak{W}^*(\Lambda_0)}$

of μ to $\mathfrak{W}^*(\Lambda_0)$ is given by the PDF $f^{\Lambda_0}(\Omega_0^*) := \frac{\mu|_{\mathfrak{W}^*(\Lambda_0)}(d\Omega_0^*)}{d^{\Lambda_0}\Omega_0^*}$, $\Omega_0^* \in \mathcal{W}^*(\Lambda_0)$, of the form

$$f^{\Lambda_0}(\Omega_0^*) = \mathbf{1}(\Omega_0^* \in \mathcal{W}_r^*(\Lambda_0)) \frac{z^{K(\Omega_0^*)}}{L(\Omega_0^*)} q^{\Lambda_0}(\Omega_0^*), \quad (2.3.1)$$

where the functional $q^{\Lambda_0}(\Omega_0^*)$ admits the following representation. \forall pair of cubes $\Lambda_0 \subset \Lambda \subset \mathbb{R}^d$,

$$q^{\Lambda_0}(\Omega_0^*) = \int_{\mathcal{W}_r^*(\mathbb{R}^d)} d\mu(\Omega_{\Lambda^c}^*) \mathbf{1}(\Omega_{\Lambda^c}^* \in \mathcal{W}_r^*(\Lambda^c)) \times \Xi^{\Lambda_0, \Omega_0^*}(\Lambda \setminus \Lambda_0 | \Omega_{\Lambda^c}^*). \quad (2.3.2)$$

Observe similarities with Eqn (2.2.9). At the same time, note the absence of the indicator α_Λ in the RHS of (2.3.2). Here, for a given (infinite) LC

$\Omega_{\Lambda^c}^* \in \mathcal{W}_r^*(\Lambda^c)$, the expression $\Xi^{\Lambda_0, \Omega_0^*}(\Lambda \setminus \Lambda_0 | \Omega_{\Lambda^c}^*)$ yields a partition function in $\Lambda \setminus \Lambda_0$, in the external field generated by $\Omega_{\Lambda^c}^*$ and in presence of a LC $\Omega_0^* \in \mathcal{W}_r^*(\Lambda_0)$:

$$\begin{aligned} \Xi^{\Lambda_0, \Omega_0^*}(\Lambda \setminus \Lambda_0 | \Omega_{\Lambda^c}^*) &= \int_{\mathcal{W}_r^*(\Lambda \setminus \Lambda_0)} d\Omega_{\Lambda \setminus \Lambda_0}^* \frac{z^{K(\Omega_{\Lambda \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda \setminus \Lambda_0}^*)} \\ &\times \mathbf{1}\left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* \vee \Omega_{\Lambda^c}^* \in \mathcal{W}_r^*(\mathbb{R}^d)\right) \\ &\times \exp \left[-h \left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^* \right) \right]. \end{aligned} \quad (2.3.3)$$

Comparing to Eqn (2.2.10) we see a difference: the integral $\int_{\mathcal{W}_r^*(\Lambda \setminus \Lambda_0)} d\Omega_{\Lambda \setminus \Lambda_0}^*$ in (2.3.3) provides a simplification. In turn, $h \left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^* \right)$ represents the energy of the concatenated LC $\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^*$ over Λ , in the external potential generated by the LC $\Omega_{\Lambda^c}^*$ over Λ^c . Formally, $h \left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^* \right)$ is defined, for $\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* \vee \Omega_{\Lambda^c}^* \in \mathcal{W}_r^*(\mathbb{R}^d)$, as the limit:

$$h \left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^* \right) = \lim_{L \rightarrow +\infty} h \left[\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda(L) \setminus \Lambda}^* \right]. \quad (2.3.4)$$

Here $\Lambda(L)$ stands for the cube $[-L, L]^{\times d}$ of side-length $2L$ centered at the origin in \mathbb{R}^d and $\Omega_{\Lambda(L) \setminus \Lambda}^*$ denotes the restriction $\Omega_{\Lambda^c}^* \Big|_{\Lambda(L) \setminus \Lambda}$ of $\Omega_{\Lambda^c}^*$ to $\Lambda(L) \setminus \Lambda$. The equations (2.3.1)–(2.3.4) are referred to as infinite-volume FK-DLR equation.

For short, a PM $\mu = \mu_{\mathbb{R}^d}$ satisfying (2.3.1)–(2.3.4) is called FK-DLR; we will also employ the term an FK-DLR probability measure (FK-DLR PM). The class of FK-DLR PMs (for a given pair of values $z \in (0, 1)$, $\beta \in (0, +\infty)$) is denoted by $\mathfrak{K}(z, \beta)$, or, briefly, \mathfrak{K} . It is straightforward that any PM $\mu \in \mathfrak{K}$ is supported by the set $\mathcal{W}_r^*(\mathbb{R}^d)$: $\mu(\mathcal{W}_r^*(\mathbb{R}^d)) = 1$. \triangle

Definition 2.6. Let $\mu \in \mathfrak{K}(z, \beta)$ be an FK-DLR PM. In this definition we associate with μ a family of integral kernels $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ where $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}(\Lambda_0)$ and $\Lambda_0 \subset \mathbb{R}^d$ is an arbitrary cube. Namely, when $\sharp \mathbf{x}_0 = \sharp \mathbf{y}_0$, we set:

$$F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) = \int_{\underline{\mathcal{W}}_r^*(\mathbf{x}_0, \mathbf{y}_0)} d\underline{\mathbb{P}}_{\mathbf{x}_0, \mathbf{y}_0}^*(\underline{\Omega}_0^*) z^{K(\underline{\Omega}_0^*)} \chi^{\Lambda_0}(\underline{\Omega}_0^*) \widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*). \quad (2.3.5)$$

In turn, the quantity $\widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*)$ admits the following integral representation involving PM μ : \forall cube $\Lambda \subset \mathbb{R}^d$ containing Λ_0 ,

$$\begin{aligned} \widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\mathbb{R}^d)} d\mu(\underline{\Omega}_{\Lambda^c}^*) \chi^{\Lambda_0}(\underline{\Omega}_{\Lambda^c}^*) \\ &\times \mathbf{1}\left((\underline{\Omega}_0^*, \underline{\Omega}_{\Lambda^c}^*) \in \underline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \Lambda^c)\right) \widehat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}(\Lambda \setminus \Lambda_0 | \underline{\Omega}_{\Lambda^c}^*). \end{aligned} \quad (2.3.6)$$

Like before, $\underline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \Lambda^c)$ denotes here the subset of the Cartesian product $\underline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0) \times \mathcal{W}_{\mathbf{r}}^*(\Lambda^c)$ formed by pairs $(\underline{\Omega}_0^*, \underline{\Omega}_{\Lambda^c}^*)$ such that \forall path $\overline{\omega}^* \in \underline{\Omega}_0^*$ and $0 \leq \bar{l} < k(\overline{\omega}^*)$ and $\forall \omega^* \in \underline{\Omega}_{\Lambda^c}^*$ and $0 \leq l < k(\omega^*)$, the Euclidean norm

$$|\overline{\omega}^*(\bar{l}\beta + \mathbf{t}) - \omega^*(l\beta + \mathbf{t})|_{\text{Eu}} > \mathbf{r} \quad \forall t \in [0, \beta].$$

Next, for a given PC $\underline{\Omega}_0^* \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0)$ over Λ_0 and LC $\underline{\Omega}_{\Lambda^c}^* \in \mathcal{W}_{\mathbf{r}}^*(\Lambda^c)$ over Λ^c such that the pair $(\underline{\Omega}_0^*, \underline{\Omega}_{\Lambda^c}^*) \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \Lambda^c)$, the expression $\widehat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}(\Lambda \setminus \Lambda_0 | \underline{\Omega}_{\Lambda^c}^*)$ is defined similarly to (2.2.11):

$$\begin{aligned} \widehat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}(\Lambda \setminus \Lambda_0 | \underline{\Omega}_{\Lambda^c}^*) &= \int_{\mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda_0)} d\Omega_{\Lambda \setminus \Lambda_0}^* \chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda_0}^*) \\ &\times \mathbf{1}\left((\underline{\Omega}_0^*, \Omega_{\Lambda \setminus \Lambda_0}^* \vee \underline{\Omega}_{\Lambda^c}^*) \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \mathbb{R}^d \setminus \Lambda)\right) \\ &\times \frac{z^{K(\Omega_{\Lambda \setminus \Lambda_0}^*)}}{L(\Omega_{\Lambda \setminus \Lambda_0}^*)} \exp \left[-h(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \underline{\Omega}_{\Lambda^c}^*) \right]. \end{aligned} \quad (2.3.7)$$

Again, $\overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \mathbb{R}^d \setminus \Lambda)$ stands for the subset in $\overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0) \times \mathcal{W}_{\mathbf{r}}^*(\mathbb{R}^d \setminus \Lambda)$ formed by pairs $(\underline{\Omega}_0^*, \underline{\Omega}_{\mathbb{R}^d \setminus \Lambda}^*)$ such that \forall path $\overline{\omega}^* \in \underline{\Omega}_0^*$ and $0 \leq \bar{l} < k(\overline{\omega}^*)$ and $\forall \omega^* \in \underline{\Omega}_{\mathbb{R}^d \setminus \Lambda}^*$ and $0 \leq l < k(\omega^*)$,

$$|\overline{\omega}^*(\bar{l}\beta + \mathbf{t}) - \omega^*(l\beta + \mathbf{t})|_{\text{Eu}} > a \quad \forall t \in [0, \beta].$$

The indicators $\chi^{\Lambda_0}(\underline{\Omega}_0^*)$, $\chi^{\Lambda_0}(\overline{\Omega}_{\Lambda^c}^*)$ and $\chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda_0}^*)$ in (2.3.5)–(2.3.7) are defined similarly to (2.2.5). Namely, for a PC $\underline{\Omega}_0^* = \{\overline{\omega}^*\} \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\mathbf{x}_0, \mathbf{y}_0)$ and a LC $\Omega_{\Lambda \setminus \Lambda_0}^* = \{\omega_x^*, x \in \mathbf{x}_{\Lambda \setminus \Lambda_0}\} \in \mathcal{W}_{\mathbf{r}}^*(\Lambda \setminus \Lambda_0)$, with $\mathbf{x}_{\Lambda \setminus \Lambda_0} \in \mathcal{C}(\Lambda \setminus \Lambda_0)$:

$$\chi^{\Lambda_0}(\underline{\Omega}_0^*) = \mathbf{1}\left(\overline{\omega}^*(\bar{l}\beta) \in \mathbb{R}^d \setminus \Lambda_0, \forall 1 \leq \bar{l} < k(\overline{\omega}^*) \quad \forall \overline{\omega}^* \in \underline{\Omega}_0^*\right),$$

and

$$\chi^{\Lambda_0}(\Omega_{\Lambda \setminus \Lambda_0}^*) = \mathbf{1}\left(\omega_x^*(l\beta) \in \mathbb{R}^d \setminus \Lambda \quad \forall l < k(\omega_x^*) \quad \forall x \in \mathbf{x}(\Omega_{\Lambda \setminus \Lambda_0}^*)\right).$$

Likewise, for a LC $\Omega_{\Lambda^c}^* = \{\omega_x^*, x \in \mathbf{x}(\Omega_{\Lambda^c}^*)\} \in \mathcal{W}^*(\Lambda^c)$:

$$\chi^{\Lambda_0}(\Omega_{\Lambda^c}^*) = \mathbf{1}\left(\omega_x^*(l\beta) \in \mathbb{R}^d \setminus \Lambda \ \forall l < k(\omega_x^*), \omega_x^* \in \Omega_{\Lambda^c}^*\right). \quad (2.3.8)$$

Finally, similarly to (2.3.4), for $(\underline{\Omega}_0^*, \Omega_{\Lambda \setminus \Lambda_0}^* \vee \Omega_{\Lambda^c}^*) \in \overline{\mathcal{W}}_{\mathbf{r}}^*(\Lambda_0, \mathbb{R}^d \setminus \Lambda)$ we set:

$$h\left(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^*\right) = \lim_{L \rightarrow +\infty} h\left[\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda(L)}^*\right]. \quad (2.3.9)$$

When $\sharp \mathbf{x}_0 \neq \sharp \mathbf{y}_0$, we set: $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) = 0$. \triangle

It is instructive to re-write the definitions (2.3.4) and (2.3.9) in line with (2.1.2), (2.1.4), (2.1.8) and (2.1.10), expressing the functionals $h\left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^*\right)$ and $h\left(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^*\right)$ in terms of energies of CCs $\Omega_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t})$, $\underline{\Omega}_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t})$ and $\Omega_{\Lambda^c}^*(\mathbf{t})$ forming t -sections of the corresponding loop and PCs, where $0 \leq \mathbf{t} \leq \beta$. Namely,

$$h\left(\Omega_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \Omega_{\Lambda^c}^*\right) = \int_0^\beta d\mathbf{t} E\left[\Omega_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) | \Omega_{\Lambda^c}^*(\mathbf{t})\right] \quad (2.3.10)$$

where

$$\begin{aligned} E\left[\Omega_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) | \Omega_{\Lambda^c}^*(\mathbf{t})\right] \\ = E\left[\Omega_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t})\right] + E\left[\Omega_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) | \Omega_{\Lambda^c}^*(\mathbf{t})\right] \end{aligned} \quad (2.3.11)$$

and

$$\begin{aligned} E\left[\underline{\Omega}_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) | \Omega_{\Lambda^c}^*(\mathbf{t})\right] \\ = E\left[\underline{\Omega}_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t})\right] + E\left[\underline{\Omega}_0^*(\mathbf{t}) \vee \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) | \Omega_{\Lambda^c}^*(\mathbf{t})\right]. \end{aligned} \quad (2.3.12)$$

In turn, finite CCs $\Omega_0^*(\mathbf{t})$, $\underline{\Omega}_0^*(\mathbf{t})$ and $\Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t})$ and an infinite CC $\Omega_{\Lambda^c}^*(\mathbf{t})$ are given by

$$\begin{aligned} \Omega_0^*(\mathbf{t}) &= \bigcup_{\omega^* \in \Omega_0^*} \bigcup_{0 \leq l < k(\omega^*)} \{\omega^*(l\beta + \mathbf{t})\}, \quad \underline{\Omega}_0^*(\mathbf{t}) = \bigcup_{\bar{\omega}^* \in \underline{\Omega}_0^*} \bigcup_{0 \leq l < k(\bar{\omega}^*)} \{\bar{\omega}^*(l\beta + \mathbf{t})\}, \\ \Omega_{\Lambda \setminus \Lambda_0}^*(\mathbf{t}) &= \bigcup_{\omega^* \in \Omega_{\Lambda \setminus \Lambda_0}^*} \bigcup_{0 \leq l < k(\omega^*)} \{\omega^*(l\beta + \mathbf{t})\} \end{aligned}$$

and

$$\begin{aligned}\Omega_{\Lambda^c}^*(\mathbf{t}) &= \bigcup_{\omega^* \in \Omega_{\Lambda^c}^*} \bigcup_{0 \leq l < k(\overline{\omega}^*)} \{\omega^*(l\beta + \mathbf{t})\} \\ &= \left\{ \omega^*(l\beta + \mathbf{t}) : \omega^* \in \Omega_{\Lambda^c}^*, 0 \leq l < k(\overline{\omega}^*) \right\}.\end{aligned}$$

Owing to the FK-DLR property of μ , the RHS in (2.3.6) does not depend on the choice of the cube $\Lambda \supset \Lambda_0$. Moreover, the kernels F^{Λ_0} satisfy the compatibility property: \forall pair of cubes $\Lambda_1 \subset \Lambda_0$,

$$\int_{\mathcal{C}(\Lambda_0 \setminus \Lambda_1)} d\mathbf{z} F^{\Lambda_0}(\mathbf{x}_1 \vee \mathbf{z}, \mathbf{y}_1 \vee \mathbf{z}) = F^{\Lambda_1}(\mathbf{x}_1, \mathbf{y}_1), \quad \mathbf{x}_1, \mathbf{y}_1 \in \mathcal{C}(\Lambda_0). \quad (2.3.13)$$

In particular,

$$\int_{\mathcal{C}(\Lambda_0)} d\mathbf{z} F^{\Lambda_0}(\mathbf{z}, \mathbf{z}) = 1. \quad \triangle \quad (2.3.14)$$

Definition 2.7. Let μ be FK-DLR and $\{F^{\Lambda_0}\}$ be the family of kernels associated with μ by Eqns (2.3.4)–(2.3.7). Given $\phi_{\Lambda_0} \in \mathcal{H}(\Lambda_0)$, introduce a trace-class operator R^{Λ_0} acting in $\mathcal{H}(\Lambda_0)$:

$$R^{\Lambda_0} \phi_{\Lambda_0}(\mathbf{x}_0) = \int_{\mathcal{C}(\Lambda_0)} d\mathbf{y}_0 F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \phi_{\Lambda_0}(\mathbf{y}_0), \quad \mathbf{x}_0 \in \mathcal{C}(\Lambda_0). \quad (2.3.15)$$

Then, according to (2.3.13)–(2.3.14),

$$\text{tr}_{\mathcal{H}(\Lambda_0 \setminus \Lambda_1)} R^{\Lambda_0} = R^{\Lambda_1}, \quad \text{tr}_{\mathcal{H}(\Lambda_0)} R^{\Lambda_0} = 1. \quad (2.3.16)$$

The family of operators R^{Λ_0} defines a linear normalized functional on the quasilocal C^* -algebra $\mathfrak{B}(\mathbb{R}^d)$ such that for $A \in \mathfrak{B}(\Lambda_0)$

$$\varphi(A) = \text{tr}_{\mathcal{H}(\Lambda_0)}(AR^{\Lambda_0}). \quad (2.3.17)$$

We call it the FK-DLR functional generated by μ ; to stress this fact we sometimes use the notation φ_μ . If in addition φ is a state (that is, the operators R^{Λ_0} are positive-definite) then we say that φ is an FK-DLR state. In this case we call the operator R^{Λ_0} an infinite-volume FK-DLR RDM. The class of FK-DLR functionals is denoted by $\mathfrak{F} = \mathfrak{F}(z, \beta)$ and its subset consisting of the FK-DLR states by $\mathfrak{F}_+ = \mathfrak{F}_+(z, \beta)$. \triangle

Before we move further, we would like to introduce a property conventionally called a ‘Ruelle bound’. This is closely related to the so-called Campbell

theorem assessing integrals of summatory functions $\Sigma_g : \Omega^* \in \mathcal{W}_r^*(\mathbb{R}^d) \mapsto \sum_{\Omega^* \subset \Omega^*} g(\Omega^*)$:

$$\int_{\mathcal{W}_r^*(\mathbb{R}^d)} \mu(d\Omega^*) \Sigma_g(\Omega^*) = \int d\Omega^* \rho(\Omega^*) g(\Omega^*) \quad (2.3.18)$$

where $\rho = \rho_\mu$ is the moment function of the RMPP μ . The Ruelle bound with a constant $\bar{\rho} = z \exp(4\beta \bar{V} R^d / r^d)$ (cf. (1.1.19)) reads

$$\rho(\Omega^*) \leq \frac{\bar{\rho}^{K(\Omega^*)}}{L(\Omega^*)} \quad (2.3.19)$$

and follows from the representation

$$\rho(\Omega^*) = \frac{z^{K(\Omega^*)}}{L(\Omega^*)} \int_{\mathcal{W}_r^*(\mathbb{R}^d)} \mu(d\Omega^*) \exp[-h(\Omega^* | \Omega^*)]. \quad (2.3.20)$$

This bound will play a rôle in arguments from [12].

2.4. Results on infinite-volume FK-DLR PMs, functionals and states. Our results about the classes \mathfrak{K} , \mathfrak{F} and \mathfrak{F}_+ are summarized in the following theorems.

Theorem 2.1. *The class $\mathfrak{K}(z, \beta)$ of FK-DLR PMs is non-empty. Moreover, the family of FK-DLR PMs μ_Λ is compact in the weak topology, and every limiting point μ for this family lies in $\mathfrak{K}(z, \beta)$. Furthermore, the family of the Gibbs states φ_Λ is compact in the w^* -topology, and every limiting point for this family gives an element from \mathfrak{F}_+ . The same is true for any family of the PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ and states $\varphi_{\Lambda|\mathbf{x}(\Lambda^c)}$ with external CCs $\mathbf{x}(\Lambda^c) \in \mathcal{C}_r(\Lambda^c)$. Consequently, the set $\mathfrak{F}_+(z, \beta)$ is non-empty. \square*

As above, in dimension two we state a result on shift-invariance to be proved in [12]. We adopt the conditions and notations from Theorem 1.2.

Theorem 2.2. *Take $d = 2$ and let μ be a PM from $\mathfrak{K}(z, \beta)$. Then the corresponding FK-DLR functional $\varphi_\mu \in \mathfrak{F}(z, \beta)$ is shift-invariant: \forall square $\Lambda_0 \subset \mathbb{R}^2$, vector $s \in \mathbb{R}^2$ and operator $A \in \mathfrak{B}(\Lambda_0)$,*

$$\varphi_\mu(A) = \varphi_\mu(\mathbf{S}(s)A), \quad A \in \mathfrak{B}(\mathbb{R}^2).$$

In terms of the corresponding infinite-volume RDMs R^{Λ_0} :

$$R^{\mathbf{S}(s)\Lambda_0} = \mathbf{U}^{\Lambda_0}(s) R^{\Lambda_0} \mathbf{U}^{\Lambda_0}(-s).$$

\square

3 Proof of Theorems 1.1 and 2.1: a compactness argument

Let us fix a cube Λ_0 of side length $2L_0$ centered at $c = (c^1, \dots, c^d)$: cf. Eqn (1.1.16). The first step in the proof is to verify that, as $\Lambda_0 \subset \Lambda$ and cube $\Lambda \nearrow \mathbb{R}^d$, the RDMK $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ (see (2.2.2)–(2.2.4), (2.2.12)–(2.2.16)) form a compact family in $C^0(\mathcal{C}_{\mathbf{r}}(\Lambda_0) \times \mathcal{C}_{\mathbf{r}}(\Lambda_0))$. (Recall, we work with pairs $(\mathbf{x}_0, \mathbf{y}_0)$ with $\sharp \mathbf{x}_0 = \sharp \mathbf{y}_0$.) Note that Cartesian product $\mathcal{C}_{\mathbf{r}}(\Lambda_0) \times \mathcal{C}_{\mathbf{r}}(\Lambda_0)$, the range of variable $(\mathbf{x}_0, \mathbf{y}_0)$, is compact. In fact, set: $v_0 = \left\lceil (2L_0)^d / \mathbf{r}^d \right\rceil$; then any CC $\mathbf{x}_0 \in \mathcal{C}_{\mathbf{r}}(\Lambda_0)$ must have $\sharp \mathbf{x}_0 \leq v_0$. As in [5], [6] and [7], it is convenient to employ the Ascoli–Arzela theorem, i.e., verify that the functions $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ are uniformly bounded and equi-continuous.

Checking uniform boundedness is straightforward: from (2.2.12)–(2.2.16) one can see that the functional $\hat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ satisfies

$$\hat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\underline{\Omega}_0^*) \leq \exp \left[4\beta K(\underline{\Omega}_0^*) \overline{V} \mathbf{R}^d / \mathbf{r}^d \right] \quad (3.1)$$

cf. (1.1.4). By using (2.2.12)–(2.2.16), we obtain that $\forall (\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{C}_{\mathbf{r}}(\Lambda_0) \times \mathcal{C}_{\mathbf{r}}(\Lambda_0)$,

$$F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \leq v_0! \left(1 \vee \sum_{k \geq 1} \overline{\rho}^k / (2\pi\beta k)^{d/2} \right)^{v_0} \quad (3.2)$$

which yields uniform boundedness in view of (1.1.19).

The argument for equi-continuity of RDMKs is based on uniform bounds upon the gradients $\nabla_x F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ and $\nabla_y F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$, for $x \in \mathbf{x}_0$, $y \in \mathbf{y}_0$. Both cases are treated in a similar fashion; for definiteness, we consider gradients $\nabla_y F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$.

It can be seen from representation (2.2.12)–(2.2.16) that there are two contributions into the gradient. The first contribution comes from varying the measure $\underline{\mathbb{P}}_{\mathbf{x}_0, \mathbf{y}_0}^*$. The second one emerges from varying the functional $\hat{q}_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\underline{\Omega}_0^*)$, more precisely, the numerator $\hat{\Xi}^{\Lambda_0, \underline{\Omega}_0^*}[\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^c)]$ in (2.2.3). In fact, it is clear that the second contribution will come out only when we vary the term $\exp \left[-h \left(\underline{\Omega}_0^* \vee \Omega_{\Lambda \setminus \Lambda_0}^* | \mathbf{x}(\Lambda^c) \right) \right]$ in (2.2.4). Of course, we are interested in variations of a chosen point $y \in \mathbf{y}_0$.

Suppose the CCs are $\mathbf{x}_0 = (x(1), \dots, x(n))$ and $\mathbf{y}_0 = (y(1), \dots, y(n))$ and the PC is $\overline{\Omega}_0^* = (\overline{\omega}^*(1), \dots, \overline{\omega}^*(n))$. Effectively, we have to analyze the gradient $\nabla_{y(j)}$ of the following expression:

$$\begin{aligned} & \int_{\overline{\mathcal{W}}_{\mathbf{r}}^*(\mathbf{x}_0, \mathbf{y}_0)} \overline{\mathbb{P}}_{\mathbf{x}_0, \mathbf{y}_0}^*(d\overline{\Omega}_0^*) \exp \left[-h \left(\overline{\Omega}_0^* | \Omega_{\Lambda \setminus \Lambda_0}^* \vee \mathbf{x}(\Lambda^c) \right) \right] \\ &= \int_{\overline{\mathcal{W}}_{\mathbf{r}}^*(\mathbf{x}_0, \mathbf{x}_0)} \overline{\mathbb{P}}_{\mathbf{x}_0, \mathbf{x}_0}^*(d\overline{\Omega}_0^*) \exp \left[-h \left(\overline{\Omega}_0^* + \overline{Z}_0^* | \Omega_{\Lambda \setminus \Lambda_0}^* \vee \mathbf{x}(\Lambda^c) \right) \right] \\ & \quad \times \prod_{1 \leq i \leq n} \exp \left\{ -|x(i) - y(i)|_{\text{Eu}}^2 / [2k(\overline{\omega}^*(i))\beta] \right\}. \end{aligned} \quad (3.3)$$

Here \overline{Z}_0^* is a collection of linear paths: $\overline{Z}_0^* = (\overline{\zeta}^*(1), \dots, \overline{\zeta}^*(n))$ where

$$\overline{\zeta}^*(i) : \mathfrak{t} \in [0, k(\overline{\omega}^*(i))\beta] \mapsto \frac{\mathfrak{t}}{k(\overline{\omega}^*(i))\beta} (y(i) - x(i)), \quad 1 \leq i \leq n.$$

The first aforementioned contribution to the gradient emerges when we differentiate the term $\exp \left\{ -|x(j) - y(j)|_{\text{Eu}}^2 / [2k(\overline{\omega}^*(j))\beta] \right\}$, the second comes from differentiating the term $\exp \left[-h \left(\overline{\Omega}_0^* + \overline{Z}_0^* | \Omega_{\Lambda \setminus \Lambda_0}^* \vee \mathbf{x}(\Lambda^c) \right) \right]$. Consequently, we obtain a uniform bound for the absolute value of the gradient

$$\begin{aligned} \left| \nabla_y F_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \right| &\leq 2\sqrt{d} L_0 v_0! \left(1 \vee \sum_{k \geq 1} \overline{\rho}^k / (2\pi\beta k)^{d/2} \right)^{v_0} \\ &\quad \times \left(\sum_{k \geq 1} \overline{\rho}^k / (2\pi\beta k)^{1+d/2} \right) \\ &\quad + \overline{V}^{(1)} v_0! \beta (\mathbb{R}^d / \mathbf{r}^d) \left(1 \vee \sum_{k \geq 1} \overline{\rho}^k / (2\pi\beta k)^{d/2} \right)^{v_0} \\ &\quad \times \left(\sum_{k \geq 1} \overline{\rho}^k / [(2\pi)^{d/2} (\beta k)^{-1+d/2}] \right) \end{aligned} \quad (3.4)$$

where we again used (1.1.4) and (1.1.19).

Hence, the family of RDMKs $\{F_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}\}$ is compact in space $C^0(\mathcal{C}_{\mathbf{r}}(\Lambda_0) \times \mathcal{C}_{\mathbf{r}}(\Lambda_0))$. Let F^{Λ_0} be a limit-point as $\Lambda \nearrow \mathbb{R}^d$. Then we have the Hilbert-Schmidt convergence

$$\lim_{k \rightarrow \infty} \int_{\mathcal{C}(\Lambda_0) \times \mathcal{C}(\Lambda_0)} d^{\Lambda_0} \mathbf{x}_0 d^{\Lambda_0} \mathbf{y}_0 \left[F_{\Lambda | \mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) - F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \right]^2 = 0.$$

Consequently, the RDM $R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ in $\mathcal{H}(\Lambda_0)$ converges to the infinite-volume RDM R^{Λ_0} determined by the kernel F^{Λ_0} , in the Hilbert-Schmidt norm:

$$\left\| R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0} - R^{\Lambda_0} \right\|_{\text{HS}} \rightarrow 0. \quad (3.5)$$

As was mentioned, applying Lemma 1 from [10] (see also Lemma 1.5 from [5]), we obtain the trace-norm convergence:

$$\left\| R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0} - R^{\Lambda_0} \right\|_{\text{tr}} \rightarrow 0. \quad (3.6)$$

Invoking a standard diagonal process implies that the sequence of states $\varphi_{\Lambda|\mathbf{x}(\Lambda^c)}$ is w^* -compact.

Alongside with the above argument, one can establish that the PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ form a compact family as $\Lambda \nearrow \mathbb{R}^d$. More precisely, we would like to show that \forall given cube Λ_0 , the family of PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ on $(\mathcal{W}_{\mathbf{r}}^*(\Lambda_0), \mathfrak{W}(\Lambda_0))$ is compact. To this end, it suffices to check that the family $\{\mu_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}\}$ is tight as the Prokhorov theorem will then guarantee compactness.

Following an argument from [5], tightness is a consequence of two facts.

(a) The reference measure $d^{\Lambda_0}\Omega_0^*$ on $\mathcal{W}_{\mathbf{r}}^*(\Lambda_0)$ (see Definition 2.1.2 (vii)) is supported by LCs with the standard continuity modulus $\sqrt{2\epsilon \ln(1/\epsilon)}$.

(b) The PDF $f_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*) = \frac{\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(d\Omega^*)}{d^{\Lambda}\Omega^*}$ (cf. (2.2.6)) is bounded from above by a constant similar to the RHS of (3.1).

As a result, the family of limit-point PMs $\{\mu^{\Lambda_0} : \Lambda_0 \subset \mathbb{R}^d\}$ has the compatibility property and therefore satisfies the assumptions of the Kolmogorov theorem. This implies that there exists a unique PM μ on $(\mathcal{W}_{\mathbf{r}}^*(\mathbb{R}^d), \mathfrak{W}(\mathbb{R}^d))$ such that the restriction of μ on the sigma-algebra $\mathfrak{W}(\Lambda_0)$ coincides with μ^{Λ_0} .

The fact that μ is an FK-DLR PM follows from the above construction. Hence, each limit-point state φ falls in class $\mathfrak{F}_+(z, \beta)$. This completes the proof of Theorems 1.1 and 2.1.

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